ELASTIC-PLASTIC ANALYSIS OF A UNIDIRECTIONAL COMPOSITE UNDER LONGITUDINAL SHEAR

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THESIS

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ABSTRACT

The failure point of a unidirectional composite subjected to longitudinal shear loading is calculated. The method of analysis is based on the finite element technique, the incremental plasticity relations of Prandtl-Reuss, and the von Mises yield criterion. The failure point for a boron-aluminum composite and a boron-epoxy composite were computed to determine the effect of the matrix material on the composite.

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LIST OF SYMBOLS

E	Modulus of Elasticity
G	Shear Modulus
S_{ij}	Macrostress
J_i	Stress Invariant
J_i'	Deviatoric Stress Invariant
U	Strain energy
ΔF	Incremental load
Y	Yield Stress
k	Yield constant
H'	Slope of the non-linear stress-strain curve
S	Constant defined by (3-19)
c	Constant defined by (2-18)
x,y,z	Co-ordinate axes
$u,v.w$	Co-ordinate displacements
Δw	Incremental displacement
A	Triangular element area
V	Volume
$d\lambda$	Proportionality constant of (2-10)
σ_{ij}	Microstress
$\bar{\sigma}$	Equivalent Stress
σ_{ij}'	Deviatoric stress component
ε_{ij}	Strain
$\bar{\varepsilon}$	Equivalent strain
ε_{ij}'	Deviatoric strain components

δ_{ij}	Kronecker delta
ν	Poisson's ratio
Δ	Dilatation
$[k]$	Element stiffness matrix
$[K]$	System stiffness matrix
$[D^E]$	Elastic stress-strain matrix
$[D^P]$	Plastic Stress-strain matrix
<>	Row matrix
{}	Column matrix
[]	Matrix

Subscripts

x,y,z	Cartesian co-ordinate axes
i,j,k	Tensor indices

Superscripts

E	Elastic analysis
P	Plastic analysis
T	Test value
$\{\}^T$	Transpose

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I INTRODUCTION

The need for material possessing a higher strength to weight ratio has become abundantly clear to modern industry. Existing materials satisfy the present high strength-low weight requirements (high performance), but composite material technology, still in its infancy promises to raise the standards of high performance even higher. Composite material technology is particularly interesting to the aerospace industry which is constantly striving to improve the performance of aeronautical and aerospace vehicles. Therefore a possible solution to the aerospace industries' quest for higher performance materials is afforded by composite materials. A composite material combines the high strength and high modulus of one material and the low-strength and low modulus of another material (or materials) to yield a material exhibiting the necessary high strength-low weight ratio desired by the aerospace industry.

A particularly useful composite is the two material fiber reinforced composite which consists of a filament embedded in a compatible matrix material. The filament possesses the high strength and high modulus properties required while the matrix, having a low strength and low modulus adds the ductility necessary in a high strength low weight ratio composite. The matrix also provides the form of the composite required for a particular structural use. A composite plate structure usually consists of many layers of composite lamina with the adjacent lamina having different filament orientations.

The composite structure is analyzed considering each lamina as an anisotropic homogeneous material, which results in the calculation of macrostresses. If a lamina is examined on an individual point by point basis, and considering the difference in the mechanical properties of the filament and the matrix, the stresses calculated are microscopic. The microscopic stresses are important in determining at which point in the composite that yield or failure occur.

To determine the yield or failure point of the composite, tensile tests are performed on the matrix and filament materials until the elastic and plastic limits are reached. The value of the limits, namely the yield stress and failure stress are microscopic stresses. To accurately determine the behavior of a composite material the macroscopic stress state must be related to the microscopic stress state. This may be accomplished using various analytical approaches. In this study the finite element technique is employed.

Several authors have considered the macrostress analysis of unidirectionally reinforced composites. Bloom and Wilson [4] have calculated the elastic microstresses for longitudinal loading. Longitudinal shear loading analysis was obtained by Adams and Doner [1], Baker and Foye [3] and Tsai, Adams, and Doner [15]. The transverse normal loading problem was considered by Foye [8], as well as Adams and Doner [2]. Lin, Salinas, and Ito [11] employed the finite element technique to solve the problem of combined loading, calculating an initial yield surface and later extending their work into the plastic region [10].

The present investigation considers the effect of longitudinal shear loading on a boron-aluminum composite and a boron-epoxy composite. The objective of this analysis is to determine the failure loads of the

two composites and to compare the effect that the two different matrix materials have on the composite's properties.

II DESCRIPTION OF PLASTICITY

This investigation considers the elastic-plastic analysis of a unidirectional reinforced composite material subjected to longitudinal shear loading. Elastic theory being a basic subject and plastic theory being a more difficult, less studied topic will be considered here and amplified in section three, while the elastic theory will only be briefly discussed in section three. This section describes some of the assumptions of plasticity while also describing the yield criterion employed in this analysis, the Prandtl-Reuss equations and the universal stress-strain law assumption.

A. YIELD CRITERION

Plasticity may be defined as that property that allows a material to deform continuously and permanently without rupture during the application of stresses that exceed those producing yield. The final distortion does not only depend on the final state of stress, but upon the sequence of stress states producing that final state. This is in contrast to elastic deformation which depends only on the final stress state.

An important assumption of the theory of plasticity of metals, based on experimental observations, is that the volume of material remains constant under plastic deformation, that is, a hydrostatic pressure does not cause yielding. This implies plastic strain is associated with distortional energy only and hence the hydrostatic component of a state of stress does not influence the point at which yielding occurs. What is needed then, is some criterion, a so called yield

criterion, that will predict yielding under a state of stress, given only the yield stress determined from a simple uniaxial tension test.

For a homogeneous, isotropic material (same properties at all points and all directions), a criterion should not allow hydrostatic stress to influence yielding. Yield must also be independent of the directions of the axes chosen to define the system. Therefore since the material is isotropic, yielding will be related only to the intensity of stress and must be a function of the invariant of the stress tensor. The invariants are the coefficients of the cubic equation, which is associated with determining the principal stresses of the stress state. If yielding is unaffected by hydrostatic stress (P) then it depends only on the deviatoric stress. This means that the criterion must be a function of the deviatoric invariants.

Before proceeding with the yield criterion discussion, some terms need explaining. Hydrostatic stress, P, is a stress that acts equally in all directions and will cause only a change of volume, recoverable on removal of the stress. This is related to the dilatation, Δ which is defined as the change of volume per unit volume. The deviatoric stress components, σ_{ij}', are obtained as the difference of the dilatation from the total stress tensor, that is

$$\sigma_{ij} = \delta_{ij}P + \sigma_{ij}'$$

or

$$\sigma_{ij}' = \sigma_{ij} - \delta_{ij} P \qquad (2\text{-}1)$$

where $P = \frac{1}{3}\sigma_{ij}$ and δ_{ij} is the Kronecker delta.

Upon expansion, equation (2-1) gives

$$\sigma_{xx}' = \sigma_{xx}-P \qquad (2\text{-}1a)$$

$$\sigma_{yy}' = \sigma_{yy}-P \qquad (2\text{-}1b)$$

$$\sigma_{zz}' = \sigma_{zz} - P \tag{2-1c}$$

$$\sigma_{xy}' = \sigma_{xy} \tag{2-1d}$$

$$\sigma_{yz}' = \sigma_{yz} \tag{2-1e}$$

$$\sigma_{zx}' = \sigma_{zx} \tag{2-1f}$$

where the hydrostatic stress, P, may be expanded to

$$P = \frac{\sigma_{ii}}{3} = (\sigma_{xx}+\sigma_{yy}+\sigma_{zz})/3 \tag{2-2}$$

where the Einstein summation convention is followed, where a repeated subscript means summation with respect to the subscript. In this study index notation as well as cartesian notation will be exchanged freely instead of the indices having the usual numerical values. The deviatoric stress invariants are also expressed in this manner.

$$J_1' = \sigma_{ii}' = \sigma_{xx}' + \sigma_{yy}' + \sigma_{zz}' \equiv 0 \tag{2-3a}$$

$$J_2' = \tfrac{1}{2}\sigma_{ij}'\sigma_{ij}' = \tfrac{1}{2}[\sigma_{xx}'^2 + \sigma_{yy}'^2 + \sigma_{zz}'^2 + 2(\sigma_{xy}'^2 + \sigma_{yz}'^2 + \sigma_{zx}'^2)] \tag{2-3b}$$

$$J_3' = \tfrac{1}{3}\sigma_{ij}'\sigma_{jk}'\sigma_{ki}' = [\sigma_{xx}'\sigma_{yy}'\sigma_{zz}' + 2\sigma_{xy}'\sigma_{yz}'\sigma_{zx}'$$

$$-\sigma_{xx}'\sigma_{yz}'^2 - \sigma_{yy}'\sigma_{zx}'^2 - \sigma_{zz}'\sigma_{xy}'^2] \tag{2-3c}$$

Various yield criterion exist that were verified experimentally and that are applicable to the problem. The von Mises yield criterion is the simplest and most convenient to use. It may be expressed as a function of the deviatoric stress invariants

$$F(J_i') = 0$$

or specifically we will use

$$J_2' = k^2 \tag{2-4}$$

where k is a constant. When $J_2' < k^2$ there is elastic behavior, that is there is no change in the plastic strain. When $J_2' = k^2$ there can be elastic or plastic strain depending upon the rate of change of J_2'.

Other derivable forms of J_2' follow in terms of the principal stresses σ_1, σ_2, σ_3.

$$2J_2' = \sigma_1'^2 + \sigma_2'^2 + \sigma_3'^2 = 2k^2 \qquad (2\text{-}5a)$$

or

$$2J_2' = \frac{1}{3}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] = 2k^2 \qquad (2\text{-}5b)$$

also

$$2J_2' = \frac{1}{3}[(\sigma_{xx}-\sigma_{yy})^2+(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2]+2[\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2] = 2k^2 \qquad (2\text{-}6)$$

The constant k may easily be solved for in terms of Y, the yield stress in simple tension, by substituting into equation (2-5b), $\sigma_1=Y$ and $\sigma_2=\sigma_3=0$

$$\sigma_1^2+\sigma_1^2=6k^2$$

or

$$2Y^2=6k^2 \text{ and } k=\frac{Y}{\sqrt{3}} \qquad (2\text{-}7)$$

Now consider the meaning of k in pure shear. Pure shear is equivalent to the stress state $\tau_{max}= \sigma_1>0$ and $\sigma_3=-\sigma_1$ with $\sigma_2=0$. Substituting into equation (2-5b) again

$$(\tau_{max})^2+(\tau_{max})^2+(-2\tau_{max})^2=6k^2$$

and (2-8)

$$\tau_{max}=k=\frac{Y}{\sqrt{3}}$$

Hence k is the shear stress, at which a point in pure shear, will yield.

The von Mises criterion implies therefore, that yielding is not dependent upon any particular stress component, equation (2-5b) shows that it depends upon a function of distortion since the differences of principal stresses are proportional to the maximum shear stresses.

It is convenient to state yet another variation of the yield criterion. From equations (2-5b) and (2-7), we obtain

$$Y=\frac{1}{\sqrt{2}}\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\}^{\frac{1}{2}}= \bar{\sigma} \qquad (2\text{-}9)$$

The scaler quantity on the right hand side may be defined as an equivalent stress, $\bar{\sigma}$, so that when $\bar{\sigma}=Y$, yielding occurs. Thus, any value of $\bar{\sigma}$ less than Y will give rise to elastic states of stress only. When $\bar{\sigma}=Y$ there may be either elastic and or plastic components, depending upon the sign of $d\bar{\sigma}$.

B. FLOW RULE

Having selected a yield criterion it is necessary to establish a relationship between the stress components and the corresponding deformations. This is usually referred to as the flow rule.

St. Venant proposed that the directions of the increments of principal plastic strains correspond to the directions of the principal stresses. Later Lévy proposed a relationship to connect the plastic strain increment to the stress in the form

$$d\lambda=\frac{d\varepsilon_{xx}^{P}-d\varepsilon_{yy}^{P}}{\sigma_{xx}-\sigma_{yy}}=\frac{d\varepsilon_{yy}^{P}-d\varepsilon_{zz}^{P}}{\sigma_{yy}-\sigma_{zz}}=\frac{d\varepsilon_{zz}^{P}-d\varepsilon_{xx}^{P}}{\sigma_{zz}-\sigma_{xx}}=\frac{d\varepsilon_{xy}^{P}}{\sigma_{xy}}=\frac{d\varepsilon_{yz}^{P}}{\sigma_{yz}}=\frac{d\varepsilon_{zx}^{P}}{\sigma_{zx}} \qquad (2\text{-}10)$$

where the superscript P denotes plastic.

The factor $d\lambda$ establishes the relation between quantities of the same differential order. From the first and third equations of (2-10),

$$d\varepsilon_{xx}^{P}-d\varepsilon_{yy}^{P}=d\lambda(\sigma_{xx}-\sigma_{yy});\; d\varepsilon_{xx}^{P}-d\varepsilon_{zz}^{P}=d\lambda(\sigma_{xx}-\sigma_{zz}). \qquad (2\text{-}11)$$

Adding the two equations of (2-11) and recalling that the volume change due to plastic strain is zero, that is, $dV^{P}=d\varepsilon_{xx}^{P}+d\varepsilon_{yy}^{P}+d\varepsilon_{zz}^{P}=0$, or, $d\varepsilon_{yy}^{P}+d\varepsilon_{zz}^{P}=-d\varepsilon_{xx}^{P}$ we obtain

$$\left.\begin{array}{l} d\varepsilon_{xx}^{P}=\frac{2}{3}d\lambda[\sigma_{xx}-\frac{1}{2}(\sigma_{yy}+\sigma_{zz})] \\ \vdots \qquad\quad \vdots \\ d\varepsilon_{xy}^{P}=d\lambda\sigma_{xy} \end{array}\right\} \quad \text{6 Equations} \qquad (2\text{-}12)$$

Similar equations may be obtained for the other components by cyclic permutation of (2-12).

These equations may also be written in deviatoric form, using

$$\begin{array}{l} \sigma'_{xx}=\sigma_{xx}-\dfrac{(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})}{3}=\frac{2}{3}[\sigma_{xx}-\frac{1}{2}(\sigma_{yy}+\sigma_{zz})] \\ \vdots \qquad \vdots \\ \sigma'_{xy}=\sigma_{xy} \end{array}$$

and comparing with equation (2-12), we obtain

$$\begin{array}{l} d\varepsilon_{xx}^{P} = d\lambda\sigma'_{xx} \\ \vdots \qquad\quad \vdots \\ d\varepsilon_{xy}^{P} = d\lambda\sigma'_{xy} \end{array}$$

or, in general form

$$d\varepsilon_{ij}^{P} = d\lambda\sigma'_{ij} \qquad (2\text{-}13)$$

In the above equations, a rigid plastic material is assumed because no mention is made of any elastic strains. For a complete statement, the equation should be written

$$d\varepsilon_{ij}^{P} = d\lambda\sigma'_{ij} \qquad (2\text{-}14)$$

where the superscript P denotes the increment of plastic strain. The total strain increment becomes

$$d\varepsilon_{ij} = d\varepsilon_{ij}^{P} + d\varepsilon_{ij}^{E} \qquad (2\text{-}15)$$

where $d\varepsilon_{ij}^{E}$ is the elastic strain increment. Now, according to the present argument, (2-14) refers to increments of the strains as we follow the deformation path of the material; we must therefore add the corresponding elastic strain increments - that is, although the plastic strain increments are, by definition, proportional to the current values

of the deviatoric stress σ_{ij}', the contribution of the elastic strain will be those corresponding to the change in the stress state as we go from two closely neighboring states of stress.

The complete statement of the stress-strain relation becomes

$$d\varepsilon_{ij}' = \sigma_{ij}' d\lambda + \frac{d\sigma_{ij}'}{2G} \tag{2-16}$$

Equation (2-16) is the Prandtl-Reuss equation for the deviatoric strain increment. The equation derives its name from Prandtl who developed the elastic and plastic strain (ε^E and ε^P) equations for plane stress and Reuss who generalized the equation, but it was Lévy who originally developed the plastic strain equation.

The rule of plastic flow in terms of principal stresses and strains also gives

$$\frac{d\varepsilon_1^P}{\sigma_1'} = \frac{d\varepsilon_2^P}{\sigma_2'} = \frac{d\varepsilon_3^P}{\sigma_3'} = d\lambda \tag{2-17}$$

C. THE UNIVERSAL STRESS-STRAIN LAW

The Universal stress-strain law is an assumption that the plastic uniaxial stress-strain relation $\sigma_x = f(d\varepsilon_x^P)$ is the same relation between equivalent stress and strain, that is $\bar{\sigma} = f(\overline{d\varepsilon^P})$ for the general stress state. Employing the universal stress-strain law equation (2-17) may be expressed in terms of equivalent (effective) strain and stress as

$$d\lambda = C\frac{\overline{d\varepsilon^P}}{\bar{\sigma}} \tag{2-18}$$

The constant C depends upon the definitions of equivalent strain and stress. The equivalent strain increment may be expressed

$$\begin{aligned}\overline{d\varepsilon^P} &= \sqrt{\tfrac{2}{3}}\{(d\varepsilon_1^P)^2 + (d\varepsilon_2^P)^2 + (d\varepsilon_3^P)^2\}^{\frac{1}{2}} \\ &= \sqrt{\tfrac{2}{3}}\{d\varepsilon_{ij}^P \, d\varepsilon_{ij}^P\}^{\frac{1}{2}}\end{aligned} \tag{2-19}$$

The numerical value of $\sqrt{\frac{2}{3}}$ has been chosen so that in uniaxial stress, (where $d\varepsilon_{yy}^{P} = d\varepsilon_{zz}^{P} = -\frac{1}{2}\, d\varepsilon_{xx}^{P}$) we have $\overline{d\varepsilon^{P}} = d\varepsilon_{xx}^{P}$. The assumption of a universal stress-strain law then gives a value of $C = {}^{3}/_{2}$. The actual proof of the numerical value of C may be found in Appendix B.

III DERIVATION OF THE STRESS-STRAIN MATRIX

The purpose of this section is to give an explicit expression for $[D^P]$ the plastic stress-strain matrix for the von Mises material. The expression obtained takes a form that can be accommodated to the finite element analysis.

The following derivations follow closely those presented by Yamada, Yoshimura and Sakurai [16]. The well known equations of elasticity, commonly called Hooke's law may be expressed in matrix form for an isotropic material as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{E}{1+\nu} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} \tag{3-1}$$

where E is Young's modulus of elasticity and ν is Poisson's ratio. The matrix equation may be expressed as

$$\{\sigma\}=E[D^E]\{\varepsilon\} \text{ or since } E=2(1+\nu)/G$$

where G is the shear modulus of elasticity

$$\{\sigma\}=2(1+\nu)G[D^E]\{\varepsilon\} \tag{3-2}$$

where $[D^E]$ is referred to as the elastic stress-strain matrix, $\{\sigma\}$ as the column matrix of stress and $\{\varepsilon\}$ as the column matrix of strain.

Hooke's law for the longitudinal shear loading case (i.e., all strains equal to zero except ε_{yz} and ε_{xz}) greatly simpifies equation (3-1) to

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{E}{1+\nu} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} \tag{3-3}$$

It is interesting to note that in the elastic range xz and yz behavior are independent. This is illustrated by equation (3-3) where the equations for σ_{yz} and σ_{zx} uncouple.

The inverse of Hooke's law, equation (3-1), may be expressed in index notation as

$$\varepsilon_{ij} = \frac{\sigma_{ij}'}{2G} + \delta_{ij}(1-2\nu)\frac{\sigma_{kk}}{3E} \tag{3-4}$$

where ε_{ij} are the components of the strain tensor. The σ_{ij}' are the components of deviatoric stress which are defined as

$$\sigma_{ij}' = \sigma_{ij} - \tfrac{1}{3}\,\delta_{ij}\sigma_{kk} \tag{3-5}$$

and explicitly expressed by equation (2-1).

Before continuing with the derivation of the stress-strain matrix and deviatoric strain components, ε_{ij}' must be explicitly stated,

$$\varepsilon_{xx}' = \varepsilon_{xx}' - e \tag{3-6a}$$

$$\varepsilon_{yy}' = \varepsilon_{yy}' - e \tag{3-6b}$$

$$\varepsilon_{zz}' = \varepsilon_{zz}' - e \tag{3-6c}$$

$$\varepsilon_{xy}' = \varepsilon_{xy}' \tag{3-6d}$$

$$\varepsilon_{yz}' = \varepsilon_{yz}' \tag{3-6e}$$

$$\varepsilon_{zx}' = \varepsilon_{zx}' \tag{3-6f}$$

where the dilatation strain is

$$e = (\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz})/3 \tag{3-6g}$$

Equations (3-6) will help better understand the Prandtl-Reuss equations for the deviatoric strain increment expressed in index notation as

$$d\varepsilon_{ij}' = \sigma_{ij}' \, d\lambda + \frac{d\sigma_{ij}'}{2G} \qquad (3\text{-}7)$$

where according to equation (2-18)

$$d\lambda = C \frac{\overline{d\varepsilon^P}}{\bar{\sigma}}$$

Letting $H' = \frac{d\bar{\sigma}}{\overline{d\varepsilon^P}}$, $d\lambda$ becomes

$$d\lambda = C \frac{d\bar{\sigma}}{\bar{\sigma}H'} \qquad (3\text{-}8)$$

and allowing the constant C to have the value three halves

$$d\lambda = \frac{3}{2} \frac{d\bar{\sigma}}{\bar{\sigma}H'} \qquad (3\text{-}9)$$

The equivalent plastic strain increment $\overline{d\varepsilon^P}$ and the equivalent stress $\bar{\sigma}$ are equal to

$$\overline{d\varepsilon^P} = \left(\tfrac{2}{3} \, d\varepsilon_{ij}^P \, d\varepsilon_{ij}^P\right)^{\frac{1}{2}} \qquad (3\text{-}10)$$

$$\bar{\sigma} = \left[\tfrac{3}{2}(\sigma_{ij}' \, \sigma_{ij}')\right]^{\frac{1}{2}} \qquad (3\text{-}11)$$

$H' = \frac{d\bar{\sigma}}{\overline{d\varepsilon^P}}$ shown in equation (3-8) is actually the slope of the stress versus plastic strain curve in the non-linear plastic region (increment of equivalent stress $(d\bar{\sigma})$/increment of equivalent plastic strain $(\overline{d\varepsilon^P})$). This is shown by figure 1, where for the plastic problem the transition must be made from the total equivalent stress-strain curve on this universal stress-strain curve to the equivalent stress-strain curve for the non-linear range.

For the derivation of the stress-strain matrix for this elastic-plastic analysis of the longitudinal shear problem the equivalent stress becomes

$$\bar{\sigma}^2 = \{3(\sigma_{yz}'^2 + \sigma_{xz}'^2)\} \qquad (3\text{-}12)$$

and the plastic strain increment $\overline{d\varepsilon^P}$ becomes

$$\overline{d\varepsilon^P} = \{\tfrac{4}{3}(d\varepsilon_{yz}^{P2} + d\varepsilon_{xz}^{P2})\}^{\frac{1}{2}} \qquad (3\text{-}13)$$

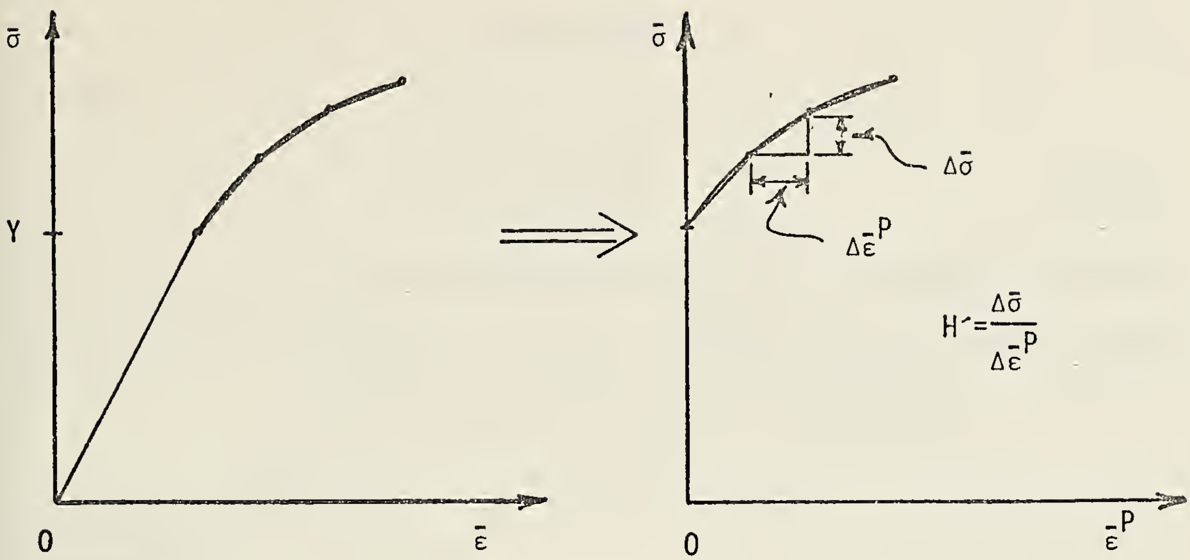


FIGURE 1. SLOPE OF THE NON-LINEAR STRESS-STRAIN CURVE.

The von Mises yield criterion is employed in the plastic analysis of the problem. It is expressed here in terms of the deviatoric stress invariant

$$2J_2' = \sigma_{ij}' \sigma_{ij}' = \frac{2}{3}\bar{\sigma}^2 \tag{3-14}$$

and in differential form

$$\sigma_{ij}' d\sigma_{ij}' = \frac{2}{3}\bar{\sigma} d\bar{\sigma} \tag{3-15}$$

Solving equation (3-9) for $d\bar{\sigma}$ and substituting into equation (3-15) we have

$$\sigma_{ij}' d\sigma_{ij}' = \frac{4}{9}\bar{\sigma}^2 H' d\lambda \tag{3-16}$$

Algebraically eliminating $d\sigma_{ij}'$ from equations (3-7) and (3-16) gives

$$2G\sigma_{ij}' (d\sigma_{ij}' - \sigma_{ij}' d\lambda) = \frac{4}{9}\bar{\sigma}^2 H' d\lambda \tag{3-17}$$

Using equation (3-11) and solving for $d\lambda$

$$d\lambda = \frac{\sigma_{ij}' \, d\varepsilon_{ij}'}{\frac{2}{3}\sigma^2 \left(\frac{H'}{3G} + 1\right)} \tag{3-18}$$

For convenience let

$$S = \tfrac{2}{3}\bar{\sigma}^2(1+H'/3G)$$

thus

$$d\lambda = \frac{\sigma_{ij}' \, d\varepsilon_{ij}'}{S} \tag{3-19}$$

The term $\sigma_{ij}' \, d\varepsilon_{ij}'$ may be simplified to $\sigma_{ij}' \, d\varepsilon_{ij}$. The proof is not very long and will be presented here rather than detailing it to an appendix.

$$\sigma_{ij}' \, d\varepsilon_{ij}' = \sigma_{xx}' \, d\varepsilon_{xx}' + \sigma_{yy}' \, d\varepsilon_{yy}' + \sigma_{zz}' \, d\varepsilon_{zz}'$$

$$+2(\sigma_{xy}' \, d\varepsilon_{xy}' + \sigma_{yz}' \, d\varepsilon_{yz}' + \sigma_{zx}' \, d\varepsilon_{zx}')$$

expanding

$$= (\tfrac{2}{3}\sigma_{xx} - (\frac{\sigma_{yy}+\sigma_{zz}}{3}))d(\frac{2\varepsilon_{xx}}{3} - (\frac{\varepsilon_{yy}+\varepsilon_{zz}}{3})) +$$

$$(\tfrac{2}{3}\sigma_{yy} - (\frac{\sigma_{xx}+\sigma_{zz}}{3}))d(\frac{2\varepsilon_{yy}}{3} - (\frac{\varepsilon_{xx}+\varepsilon_{zz}}{3})) +$$

$$(\tfrac{2}{3}\sigma_{zz} - (\frac{\sigma_{yy}+\sigma_{zz}}{3}))d(\frac{2\varepsilon_{zz}}{3} - (\frac{\varepsilon_{xx}+\varepsilon_{yy}}{3})) +$$

$$2(\sigma_{xy} \, d\varepsilon_{xy} + \sigma_{yz} \, d\varepsilon_{yz} + \sigma_{zx} \, d\varepsilon_{zx})$$

which simplifies to

$$= \tfrac{1}{3}[d\varepsilon_{xx}(2\sigma_{xx}-(\sigma_{yy}+\sigma_{zz})) + d\varepsilon_{yy}(2\sigma_{yy}-(\sigma_{xx}+\sigma_{zz}))$$

$$+ d\varepsilon_{zz}(2\sigma_{zz}-(\sigma_{xx}+\sigma_{yy}))] + 2(\sigma_{xy}d\varepsilon_{xy}+\sigma_{yz}d\varepsilon_{yz}$$

$$+ \sigma_{zx}d\varepsilon_{zx})$$

The term $\sigma_{ij}' \, d\varepsilon_{ij}'$ may be expressed as

$$\sigma_{ij}' \, d\varepsilon_{ij} = (\tfrac{2}{3}\sigma_{xx} - \frac{(\sigma_{yy}+\sigma_{zz})}{3})d\varepsilon_{xx} +$$

$$(\tfrac{2}{3}\sigma_{yy}-\frac{(\sigma_{xx}+\sigma_{zz})}{3})d\varepsilon_{yy}+(\tfrac{2}{3}\sigma_{zz} - \frac{(\sigma_{xx}+\sigma_{yy})}{3})d\varepsilon_{zz}$$

$$+2(\sigma_{xy} \, d\varepsilon_{xy} + \sigma_{yz} \, d\varepsilon_{yz} + \sigma_{zx} \, d\varepsilon_{zx})$$

Therefore

$$\sigma_{ij}' \, d\varepsilon_{ij}' = \sigma_{ij}' \, d\varepsilon_{ij}$$

Now the deviatoric stress increment $d\sigma_{ij}'$ from equations (3-16), (3-17), and (3-19) is equal to

$$d\sigma_{ij}' = 2G(d\varepsilon_{ij}' - \sigma_{ij} \frac{\sigma_{kl} \, d\varepsilon_{kl}}{S})$$

$$= 2G(d\varepsilon_{ij} - \delta_{ij} \frac{d\varepsilon_{kk}}{3} - \sigma_{ij}' \frac{\sigma_{kl}' d\varepsilon_{kl}}{S}) \qquad (3\text{-}20)$$

which was obtained by recalling equations (3-6) which may be expressed in incremental index notation as

$$d\varepsilon_{ij}' = d\varepsilon_{ij} - \delta_{ij} \frac{d\varepsilon_{kk}}{3} \qquad (3\text{-}21)$$

and substituting $d\varepsilon_{ij}'$ and $d\lambda$ into equation (3-7) and solving for $d\sigma_{ij}'$ equation (3-20) was derived.

By definition the total stress increment $d\sigma_{ij}$ is

$$d\sigma_{ij} = d\sigma_{ij}' + \frac{E}{3(1-2\nu)} \delta_{ij} \, d\varepsilon_{kk} =$$

$$d\sigma_{ij}' + \frac{2(1+\nu)}{3(1-2\nu)} G \, \delta_{ij} \, d\varepsilon_{kk} \qquad (3\text{-}22)$$

Finally substituting (3-20) into (3-22)

$$d\sigma_{ij} = 2G\left(d\varepsilon_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} d\varepsilon_{kk} - \sigma_{ij}' \frac{\sigma_{kl} d\varepsilon_{kl}}{S}\right) \qquad (3\text{-}23)$$

which may be expressed in matrix notation as

$$\{d\sigma\} = E[D^P]\{d\varepsilon\} = 2(1+\nu)G[D^P]\{d\varepsilon\} \qquad (3\text{-}24)$$

Mention should be made here of the particular notation for the elastic and plastic stress-strain equations. The elastic equation is expressed in the usual tensor form while the plastic equation is expressed in differential form. The plastic problem is point dependent, while the

elastic problem is not. The value of $[D^P]$ is a function of what point on the plastic stress-strain curve the calculation is being made while the value of $[D^E]$ is a constant anywhere on the elastic curve.

The elastic and plastic equations are similar enough to allow the replacement of $[D^E]$ for a yielded material by $[D^P]$. Explicitly $[D^P]$ becomes

$$D^P = \frac{\nu}{1+\nu}\begin{bmatrix} \frac{1-\nu}{1-2\nu} - \frac{\sigma_{xx}'^2}{S} & \frac{\nu}{1-2\nu} - \frac{\sigma_{xx}'\sigma_{yy}'}{S} & \frac{\nu}{1-2\nu} - \frac{\sigma_{xx}'\sigma_{zz}'}{S} & -\frac{\sigma_{xx}'\sigma_{xy}}{S} & -\frac{\sigma_{xx}'\sigma_{yz}}{S} & -\frac{\sigma_{xx}'\sigma_{zx}}{S} \\ & \frac{1-\nu}{1-2\nu} - \frac{\sigma_{yy}'^2}{S} & \frac{\nu}{1-2\nu} - \frac{\sigma_{yy}'\sigma_{zz}'}{S} & -\frac{\sigma_{yy}'\sigma_{xy}}{S} & -\frac{\sigma_{yy}'\sigma_{yz}}{S} & -\frac{\sigma_{yy}'\sigma_{zx}}{S} \\ & & \frac{1-\nu}{1-2\nu} - \frac{\sigma_{zz}'^2}{S} & -\frac{\sigma_{zz}'\sigma_{xy}}{S} & -\frac{\sigma_{zz}'\sigma_{yz}}{S} & -\frac{\sigma_{zz}'\sigma_{zx}}{S} \\ & & & \frac{1}{2} - \frac{\sigma_{xy}^2}{S} & -\frac{\sigma_{xy}\sigma_{yz}}{S} & -\frac{\sigma_{xy}\sigma_{zx}}{S} \\ & \text{Symmetry} & & & \frac{1}{2} - \frac{\sigma_{yz}^2}{S} & -\frac{\sigma_{yz}\sigma_{zx}}{S} \\ & & & & & \frac{1}{2} - \frac{\sigma_{zx}^2}{S} \end{bmatrix} \qquad (3\text{-}25)$$

For the longitudinal shear loading case $[D^P]$ becomes

$$D^P = \frac{\nu}{1+\nu}\begin{bmatrix} \frac{1}{2} - \frac{\sigma_{yz}^2}{S} & -\frac{\sigma_{yz}\sigma_{zx}}{S} \\ -\frac{\sigma_{yz}\sigma_{zx}}{S} & \frac{1}{2} - \frac{\sigma_{zx}^2}{S} \end{bmatrix} \qquad (3\text{-}26)$$

The plastic stress-strain matrix $[D^P]$, unlike the elastic stress-strain matrix $[D^E]$, does not uncouple to give independent xz and yz behavior. The matrix $[D^P]$ is the non-linear, point dependent, plastic stress-strain matrix relating incremental stress and strain.

The next section describes how $[D^E]$ and $[D^P]$ fit into the scheme of the analysis to determine the failure point of the composite.

IV METHOD OF ANALYSIS

The objective of this analysis is to determine the failure point of a unidirectional fiber reinforced composite material subjected to longitudinal shear loading and to determine the effect that the matrix material has on the overall properties of the composite.

A. THE NON-LINEAR PLASTIC PROBLEM

The method for the plastic analysis described here was developed by Yamada, Yoshimura and Sakurai [16] for plane stress and modified for the longitudinal shear loading case. Briefly the analysis solves the incremental force-displacement equilibrium equation

$$[K^P]\ \{dw\} = \{dF\} \qquad (4\text{-}1)$$

where $[K^P]$ is the stiffness matrix, $\{dw\}$ is the differential displacement in the Z-direction and $\{dF\}$ is the differential force vector in the Z-direction. Equation (4-1) is obtained by finite element discretization of the displacement vector $\{w\}$, this may be written

$$[K^P(w)]\{dw\} = \{dF\} \qquad (4\text{-}2)$$

and solving for $\{dw\}$

$$\{dw\} = [K^P(w)]^{-1}\{dF\} \qquad (4\text{-}3)$$

In the above problem the differential displacement is calculated on the basis of the preceding displacement, that is, the non-linear problem is solved by incremental theory,

$$\{\Delta w_i\} = [K^P(w_{i-1})]^{-1}\ \{\Delta F\} \qquad (4\text{-}4)$$

The stiffness matrix $[K^P]$ in the preceding discussion is a function of the displacement vecter $\{w\}$ in the plastic analysis only. The stiffness matrix for the elastic analysis is a function of two independent

elastic constants and the equation {dw} = [K] {dF} may be integrated to give {w} = [K^E] {F}. Hence the linear elastic problem can be done in one analysis while the non-linear plastic problem requires repeated incremental analysis. That is, the non-linear problem is approximated by a succession of linear problems. This is illustrated in figure 2.

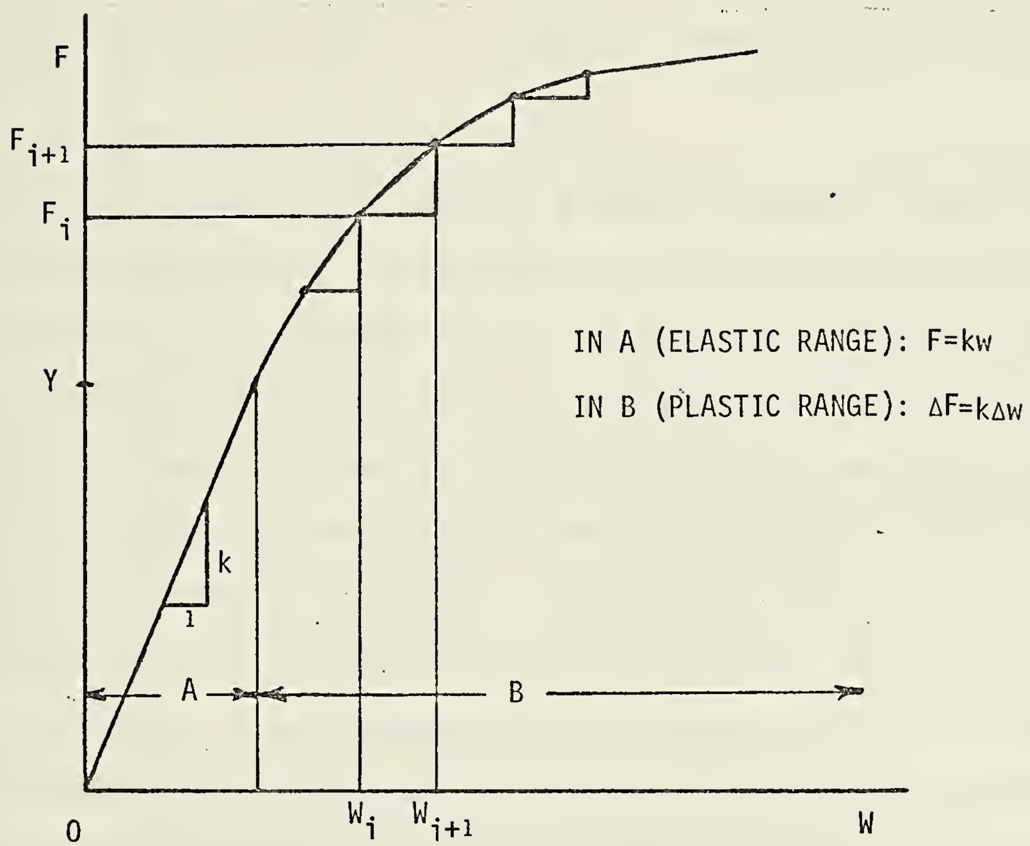


FIGURE 2. THE ELASTIC-PLASTIC PROBLEM

The present discussion is limited to general terms, but a more detailed presentation will be related in Section 5.

The problem considered in this analysis is the longitudinal shear loading case. The edge x=a is given a uniform unit displacement in the positive Z-direction as shown in figure 5. The resulting macrostresses S_{xz} and S_{yz} and microstresses σ_{xz} and σ_{yz} are calculated by the finite element method using a linear strain triangle (6 degrees of freedom). Felippa [6] presents a derivation of the linear strain triangle for in-plane behavior and the derivation of the linear strain triangles for longitudinal shear behavior is presented in Appendix A.

B. DESCRIPTION OF THE FILAMENT COMPOSITE

The microscopic stress state in a composite material depends upon the arrangement of the filament within the matrix and the applied loading. To simplify the present analysis, we consider the special case in which the filaments are arranged in a doubly periodic rectangular array. This array of filaments in the matrix may be divided into basic unit cells of dimensions 2ax2b, to facilitate ease of calculation. This arrangement is illustrated in figure 3. The arrangement illustrated in figure 4 with square cell a=b=0.0052 in., and filament radius c=0.004 in., results in a filament volume of approximately 50 percent of the total volume. The symmetry of the basic unit cell allows the analysis to consider only one quadrant of the basic unit cell as shown in figure 4. The quadrant considered has dimensions axb with a filament of radius c. It is also assumed that the filament and matrix material are homogeneous and isotropic and that a perfect bond exists between the filaments and the matrix material, i.e., there is displacement continuity along the interface.

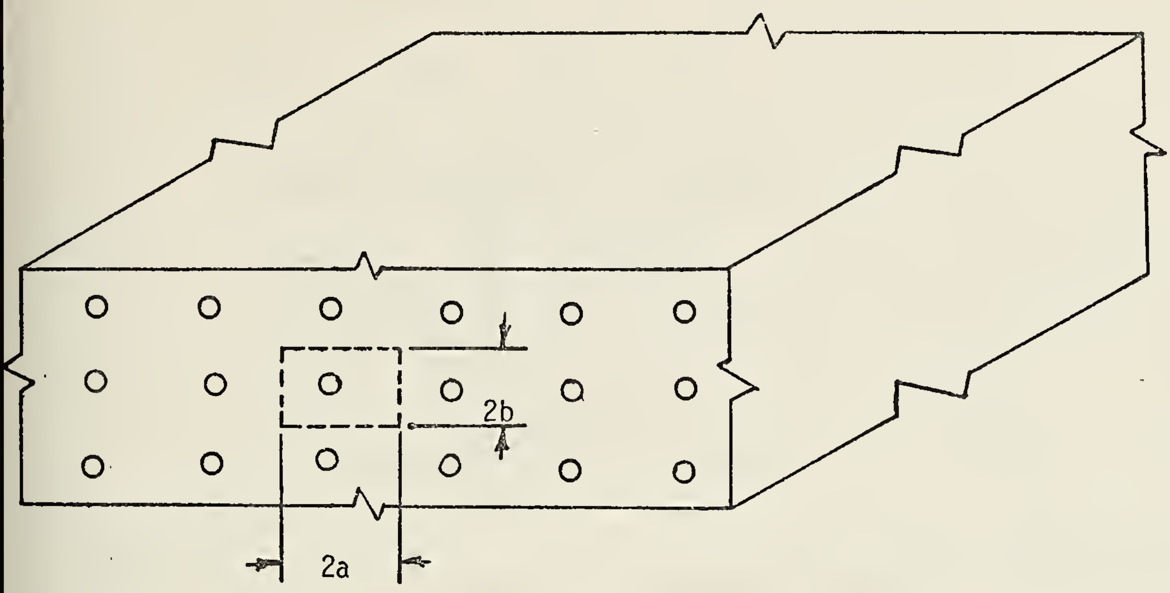


FIGURE 3. RECTANGULAR ARRAY OF FILAMENTS AND A BASIC UNIT.

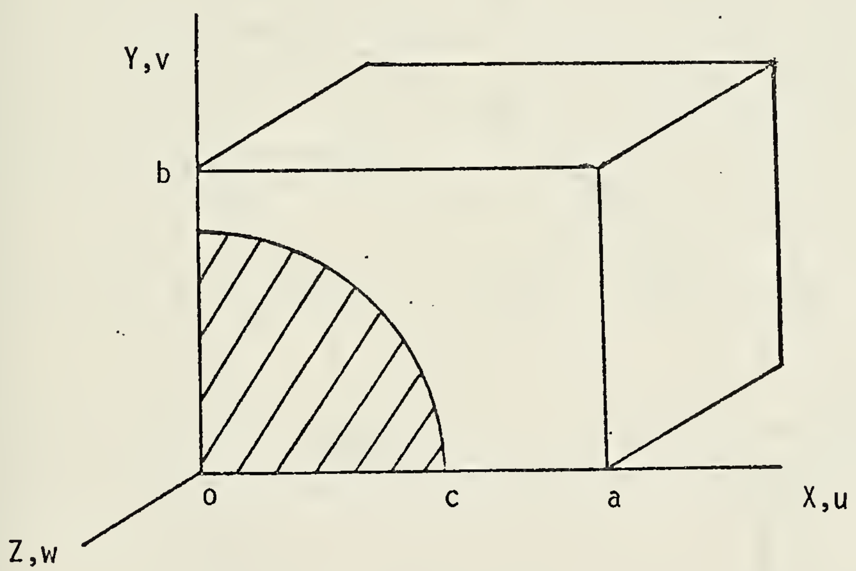


FIGURE 4. A QUADRANT OF THE BASIC UNIT.

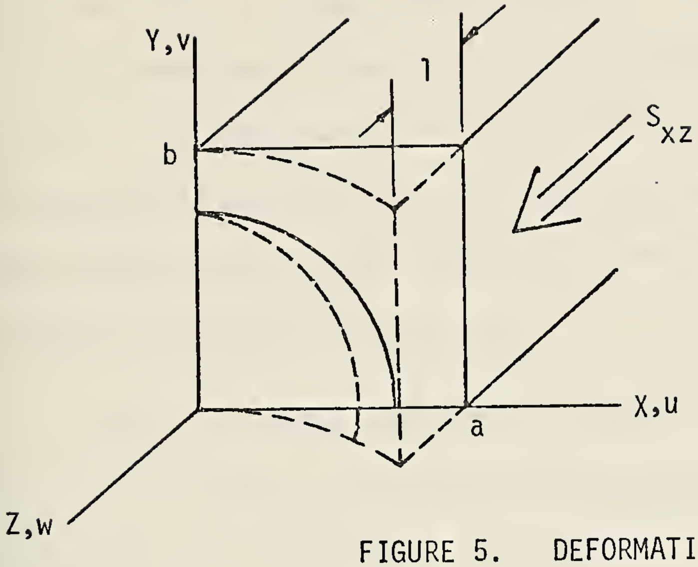


FIGURE 5. DEFORMATION STATE

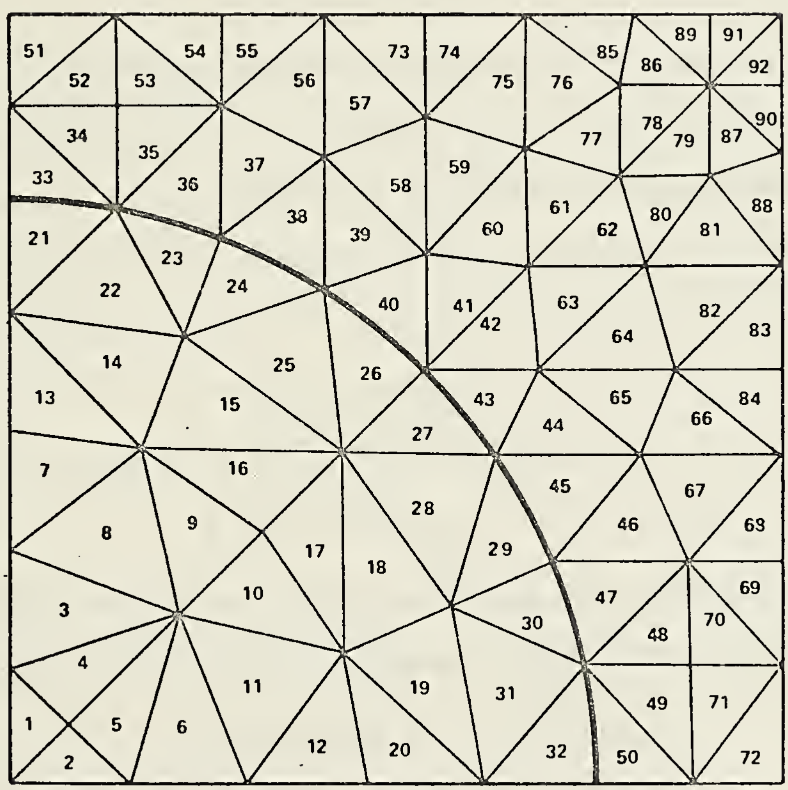


Figure 6. FINITE ELEMENT MODEL FOR CIRCULAR FILAMENT CROSS-SECTION

The finite element model used is shown in figure 6. The model is divided into 92 elements and a total of 215 nodes or degrees of freedom. The 92 element model was chosen over a 54 element model and a 200 element model. Previous analysis of Lin et al [11] indicated that the 54 element model did not yield results of sufficient accuracy, while the 200 element model gave very accurate results, but were not commensurate with the amount of computer time required.

C. DERIVATION OF THE EQUILIBRIUM EQUATION

In the elastic region the resulting strains obey the linear Hookian stress-strain relation,

$$\underset{6x1}{\{\sigma\}} = \underset{6x6}{[D^E]}\ \underset{6x1}{\{\varepsilon^E\}} \tag{4-5}$$

where $\{\sigma\}$ is the column vector of stresses, $[D^E]$ is the elastic stress-strain matrix and $\{\varepsilon^E\}$ is the column vector of elastic strains. The explicit form of the symmetric matrix $[D^E]$ is given by equation (3-1). The stiffness matrix $[k]$ for the elastic elements may be derived from energy considerations. If

$$dU = \sigma_{ij}\, d\, \varepsilon_{ij},$$

then the principle of virtual work shows that,

$$[k]\ \{w\} = \partial U/_{\partial\{w\}} \tag{4-6}$$

The particular expression for the stiffness matrix of the longitudinal shear problem for the elastic problem is obtained as follows:

$$U = \iiint_V \{\sigma\}^T\{d\varepsilon\}dV = \tfrac{1}{2}\iiint_V \{\sigma\}^T\{\varepsilon\}dV \tag{4-7}$$

where we used the fact that stress is linearly proportional to strain, where the $\{\sigma\}^T$ is the transpose of $\{\sigma\}$. Using equation (4-5) and (4-7),

we obtain

$$U = \frac{1}{2}\iiint_V \{\varepsilon^E\}^T[D^E]^T\{\varepsilon^E\}dV \tag{4-8}$$

The strain-displacement relation may be expressed as

$$\{\varepsilon\} = [B]\{w\} \tag{4-9}$$

where [B] is the coefficient matrix and {w} is the displacement vector in the Z-direction. Substituting this relation into the previous energy equation (4-8)

$$U = \{w\}^T\frac{1}{2}\iint_A [B]^T[D^E]^T[B]\ t\ dA\{w\} \tag{4-10}$$

where t is the uniform thickness. From equation (4-10) the elastic stiffness matrix for an element becomes

$$[k^E] = \iint_A [B]^T[D^E]^T[B]tdA \tag{4-11}$$

where A is the area of the triangular element. Similarly for the plastic element, the differential stress-strain relation is

$$\underset{6x1}{\{d\sigma\}} = \underset{6x6}{[D^P]}\ \underset{6x1}{\{d\varepsilon^P\}} \tag{4-12}$$

where the explicit form of $[D^P]$ is given by equation (3-25). In this case the element stiffness matrix is

$$[k^P] = \iint [B]^T[D^P]^T[B]tdA \tag{4-13}$$

The stiffness matrix $[k^E]$ for the elastic elements and $[k^P]$ for the plastic elements are assembled to form [K], the stiffness matrix of the whole body. The system stiffness matrix relates the incremental load $\{\Delta F\}$ to the incremental nodal displacements $\{\Delta w\}$ for the elastic-plastic analysis

$$\{\Delta F\} = [K]\{\Delta w\} \tag{4-14}$$

Equation (4-14) is basically how the plastic, non-linear problem is approached. The non-linear problem is linearized by taking very small

finite steps (increments) along the plastic portion of the stress-strain curve. In this way the equivalent stress for each element will not deviate during succeeding steps from the actual stress-strain curve to any significant amount. The following discussion supplies a little more detail to this sketch.

D. SOLUTION TECHNIQUE

Using the incremental load technique the elastic-plastic problem is approached as follows. First the elastic problem is done in one step. For a test shear displacement (unity), solve [K]{w} = {F}. This equation holds for the elastic range only. Calculate the elastic displacements at the nodes and then the elastic strains, stresses, and equivalent stresses $\bar{\sigma}^E$ for each element. Determine the maximum $\bar{\sigma}^E$ and scale all elastic values using a scale factor of $r = Y/\bar{\sigma}^E_{max}$, where Y is the yield stress of the material resulting from a uniaxial tension test. This is done to bring the first element or elements to yield. Next calculate the $[D^P]$ and resulting $[k^P]$ for the yielded element or elements and replace the previous elastic matrices $[D^E]$ and $[k^E]$, with the new matrices in the system stiffness matrix [K]. Next a test increment $\{\Delta w^T\}$ of the shear displacement is chosen (where the superscript T inside the brackets denote the test increment). The strain increment $\Delta\varepsilon_{ij}^T$ and the stress increments $\Delta\sigma_{ij}^T$ are calculated at each element after solving for $\{\Delta w^T\}$. Now scale all elements remaining in the elastic state by the minimum of

$$R = \frac{\Gamma + (\Gamma^2 + 4(\overline{\Delta\sigma}^{\;T})^2\,(Y^2 - \bar{\sigma}^2))^{\frac{1}{2}}}{2(\overline{\Delta\sigma}^{\;T})^2} \tag{4-15}$$

where

$$\Gamma = (\overline{\Delta\sigma}^{\;T})^2 - 2\bar{\sigma}\Delta\bar{\sigma}^T - (\Delta\bar{\sigma}^T)^2 \tag{4-16}$$

$$\Delta\sigma^{T} = \left(\frac{3}{2}\,\Delta\sigma_{ij}'^{T}\,\Delta\sigma_{ij}'^{T}\right)^{\frac{1}{2}} \qquad (4\text{-}17)$$

R is the scale factor that brings the next elastic element to yield (see Appendix B for derivation). Also, $\bar{\sigma}$ is the present equivalent stress of the elastic element and $\Delta\bar{\sigma}^{T}$ denotes the increment of $\bar{\sigma}$ induced by $\{\Delta w^{T}\}$.

A fine point should be made clear before proceeding, $\overline{\Delta\sigma}^{T}$ and $\Delta\bar{\sigma}^{T}$ are not the same quantities. A significant difference does exist between the change in effective (or equivalent) stress ($\Delta\bar{\sigma}^{T}$) and the effective change in stress ($\overline{\Delta\sigma}^{T}$). The change in effective stress is the increment of equivalent stress ($\bar{\sigma}$) caused by an incremental increase in the loading. The effective change in stress is that portion of the stress increment that actually produces a change in the plasticity of an element.

At this point in the procedure the nodal displacements, strain and stress increments are multiplied by the minimum value of R and added to the existing values. Next the equivalent stress $\bar{\sigma}$, given by equations (2-9) and (3-11) and the incremental equivalent plastic strain, $\overline{\Delta\varepsilon^{P}}$ are calculated for each element.

$$\overline{\Delta\varepsilon^{P}} = \frac{\sigma_{ij}'\,\Delta\varepsilon_{ij}}{\bar{\sigma}(1+H'/_{3G})} \qquad (4\text{-}18)$$

If the value of the $\overline{\Delta\varepsilon^{P}}$ is positive, the analysis returns to the beginning of the next plastic cycle by calculating $[D^{P}]$ and $[k^{P}]$ for the next yielded element. If $\overline{\Delta\varepsilon^{P}}$ is negative the computation is stopped. This is included here because large plastic strain are possible without any practical increase in the load after the plastic region has expanded tremendously. This gives an indication that the analysis is very close to the collapse load of limit analysis. The problem may terminate in a very

natural mode when all the cycles are complete or when the incremental equivalent plastic strain, $\overline{\Delta\varepsilon^P}$ goes negative. These two modes of failure are illustrated in figure 7. The next section describes the actual procedures in much more detail.

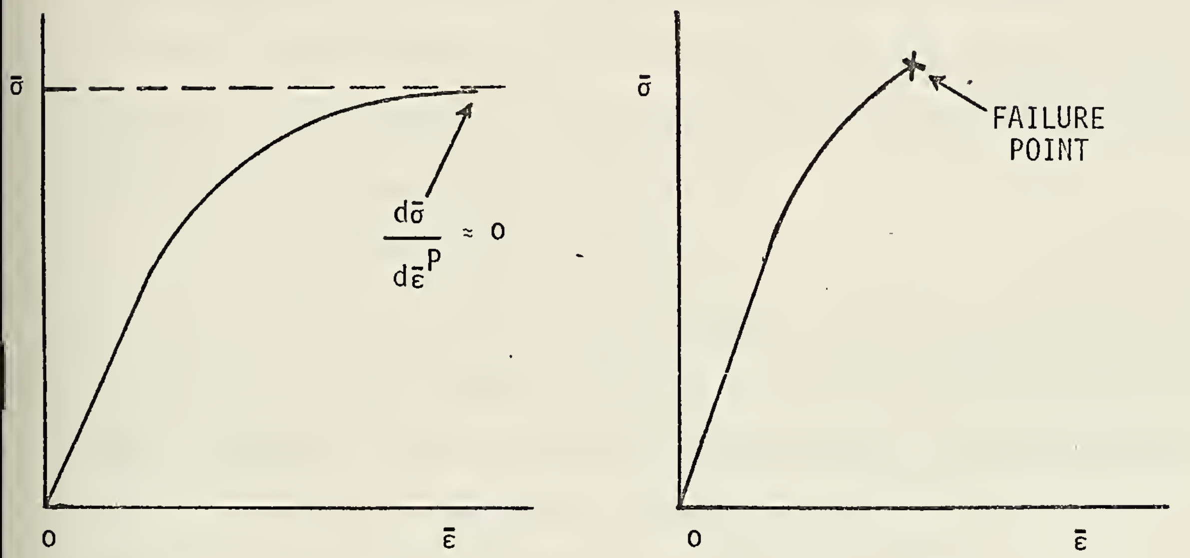


FIGURE 7. MODES OF FAILURE

V PROCEDURES OF CALCULATION

This section provides a much more detailed explanation of the procedures of calculation than does the Method of Analysis section. The procedures of calculation of this analysis was suggested by Yamada et al [16] and modified for the longitudinal shear loading problem. The analysis was constructed to be able to perform calculations for both the xz and yz shear loading problems, but will only be described for the xz shear problem. The xz shear problem was the one actually considered by this investigation. The procedure follows.

For the elastic analysis (Kw=F),

1. Apply a uniform unit displacement to the x=a face, and calculate the resulting displacements, strain, and stresses for the elastic behavior (system stiffness uses elastic stress-strain relations here). Figure 5 illustrates the uniform unit displacement given to the basic unit cell (first quadrant only).

2. Scale the elastic values so that the element with the maximum equivalent stress ($\bar{\sigma}^T_{max}$) is at initial yield (Y), i.e., adjustment of input displacement so that a single point is at yield. The scale factor r^E is

$$r^E = Y/\bar{\sigma}^T_{max} \tag{5-1}$$

and the nodal displacement vector $\{w^E\}$ and the nodal load vector $\{F^E\}$ at the initial yield point are given by

$$\{w^E\} = r^E\{w^T\} \tag{5-2}$$

$$\{F^E\} = r^E\{F^T\} \tag{5-3}$$

This terminates the elastic analysis. For the plastic analysis that follows we proceed in cycles or increments. The procedure for the plastic analysis is one cycle that is repeated over and over until one point or element of the composite has failed (termination of the problem may end in other ways, see section four and actual computer program). The loading for the plastic problem may mean either displacement or force loading, but for this particular study the displacement loading was employed. Displacement loading is used because we are interested in achieving the shear deformation state. Thus we use equation (4-4) for this problem ($\{dw_i\} = [K(w_{i-1})]^{-1}\{dF\}$).

3. Calculate $[D^P]$ and $[k^P]$ for the post yield elements; usually only one element will reach yield in the first cycle. However since elements close to yield will be carried into yield during the next increment of displacement, any element that is within .995 of yield is considered to be at yield. The value .995 is completely arbitrary; it depends on the accuracy that the user wishes to obtain.

4. Modify the system stiffness matrix to reflect the stiffness of the elements that have yielded.

5. Choose a test increment Δw^T of displacement.

6. Solve $\{dF^T\} = [K(w_{i-1})]\ \{dw_i^T\}$ and then calculate the strain increment $\{\Delta\varepsilon\}$ and the stress increment $\{\Delta\sigma\}$.

7. Calculate the scale factor R (equation 4-15) for each remaining elastic element to reach yield.

8. Determine the minimum R. The displacement increment, $R_{min}\{\Delta w^T\}$ is sufficient to yield that element having the minimum R and place it in the plastic range.

9. Multiply the strain increment and stress increment calculated in step six by R_{min} and add the resulting del (Δ) quantities at the nodes to the previous nodal values, i.e., $\{w\}_i=\{w\}_{i-1}+\{\Delta w\}$. Store the results.

10. Calculate and store the new equivalent stress for each element.

11. Calculate the increment of plastic strain for each post yield element. The stiffness of each plastic element constantly changes as the element progresses into the plastic range.

$$\overline{\Delta\varepsilon^P} = \frac{\sigma_{ij}' \, \Delta\varepsilon_{ij}}{\bar{\sigma}(1+H'/_{3G})} \tag{4-18}$$

For this analysis
$$\overline{\Delta\varepsilon^P} = \frac{(\sigma_{yz} \, \Delta\varepsilon_{yz}+\sigma_{zx} \, \Delta\varepsilon_{zx})}{\bar{\sigma}(1+H'/_{3G})} \tag{5-4}$$

12. Check that $\overline{\Delta\varepsilon^P}$ is positive. If $\overline{\Delta\varepsilon^P}$ is positive return to step three. If $\overline{\Delta\varepsilon^P}$ is negative stop computation. This is included because large plastic strains are possible without any practical increase in the load after the plastic region has expanded tremendously. This is one way of ending the problem, the other way of course is a natural ending. A natural ending occurs when the number of plastic cycles requested are run or where some of the other checks built into the program terminate the problem (see computer program in Appendix C).

VI RESULTS, CONCLUSIONS AND RECOMMENDATIONS

A. RESULTS

The computer program for this elastic-plastic analysis of the longitudinal reinforced composite loaded in the xz shear mode was run until some point in the matrix reached its ultimate strength. The properties for the matrix material and for the boron filament used in the analysis are given in table 1.

In the analysis the first point (element) to yield was calculated before the ultimate breaking point was located. The initial yield area for boron-epoxy and boron-aluminum composites occurred at element 50 as shown in figures 8 and 9 respectively. Bordering the filament-matrix interface, this point (element 50) lies in the matrix material on the lower horizontal face of the model. The filament point in both composites that obtains the highest equivalent stress at the time of initial yield is element 32, which is the filament element adjacent to matrix yield element 50, (see figure 6).

Table 1. Material Properties

Property	Boron	Aluminum	Epoxy*
Young's Modulus	60×10^6 psi	10×10^6 psi	$.5\times10^6$ psi
Poisson's Ratio	0.20	0.30	0.31
Elastic limit	480,000 psi	34,600 psi	3000 psi
Ultimate Strength	480,000 psi	42,500 psi	4200 psi
N	-	38.3	37.17
B	-	46,300 psi	5088 psi

* Tensile Properties for Epoxy

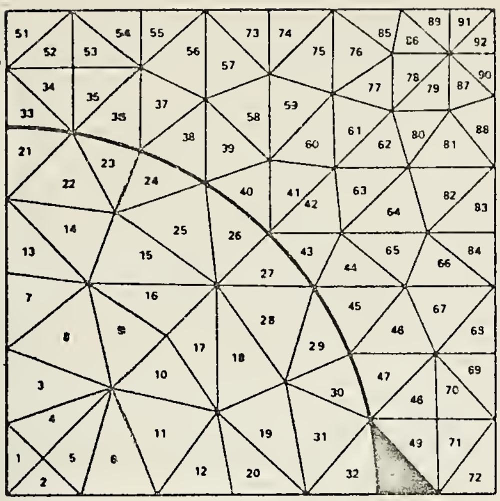
When the analysis finally reaches the breaking point for both composites, some interesting results have occurred. In the boron-epoxy composite element 50 becomes the first matrix element to fail and element 32 is still the highest stressed filament element, but in the boron-aluminum composite the initial yield element 50 is overtaken by element 70 to become the first failure element and element 32 is overtaken by element 31 to become the highest stressed filament element at the time of failure. The progression from initial yield to failure for the boron-epoxy and boron-aluminum composites is shown in figures 8 and 9 respectively.

The results of the initial yield analysis are presented in table 2. These are the results associated with the elastic limit analysis. The boron-aluminum composite has a macro shear stress S_{xz}=13,372 psi at its elastic limit; and the boron-epoxy composite has a macro stress of S_{xz}=1044 psi at its elastic limit.

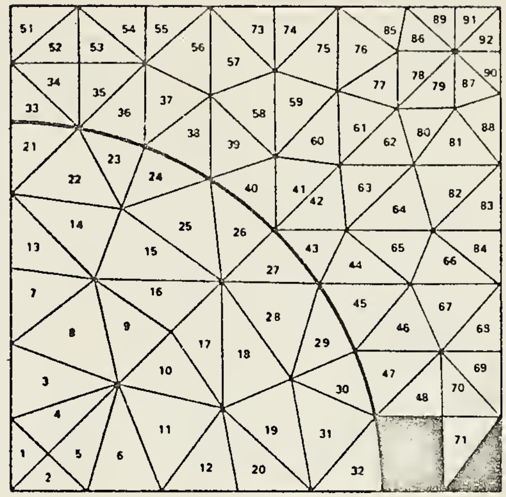
Table 2. Results at Initial Yield

	Boron-aluminum	Boron-epoxy
$\sigma_{xz_{50}}$	19,953 psi	1729 psi
$\bar{\sigma}_{max_{50}}$	34,600 psi	3000 psi
$\bar{\sigma}_{max_{32}}$	35,288 psi	3070 psi
S_{xz}	13,372 psi	1044 psi
S_{yz}	240 psi	22 psi

We note that the maximum stresses in the filament are but a small fraction of the elastic limit stress (480,000 psi) of the filament material, that is, the filament is not being used efficiently. Thus, designing to the



1. INITIAL YIELD



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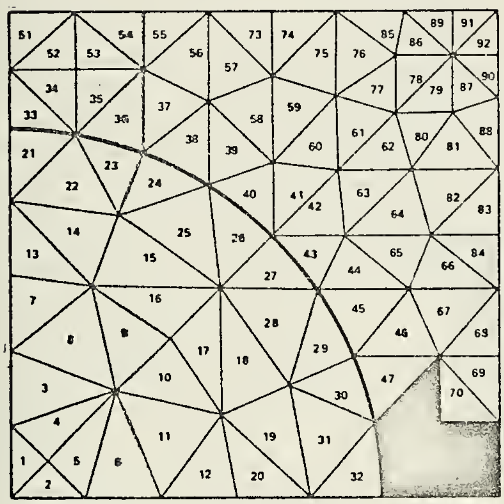

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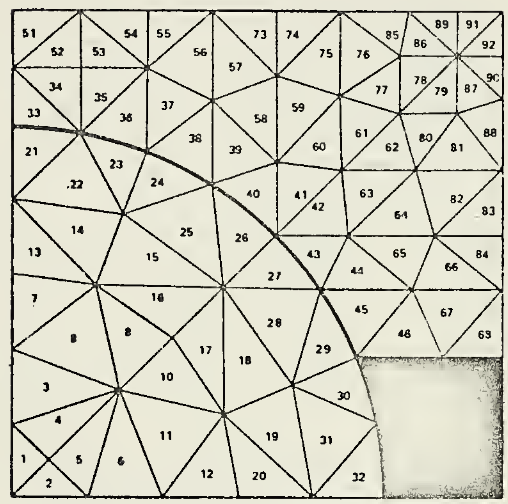

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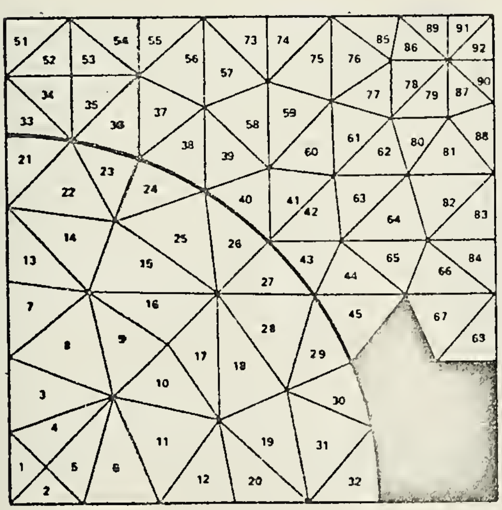

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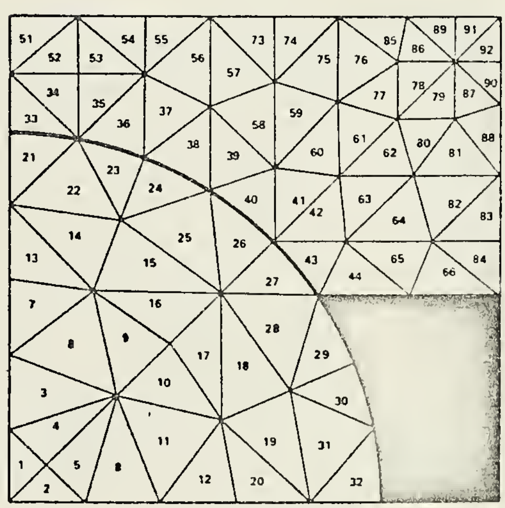


6. FAILURE

Figure 8. PROPAGATION OF PLASTIC ENCLAVE FOR BORON-EPOXY

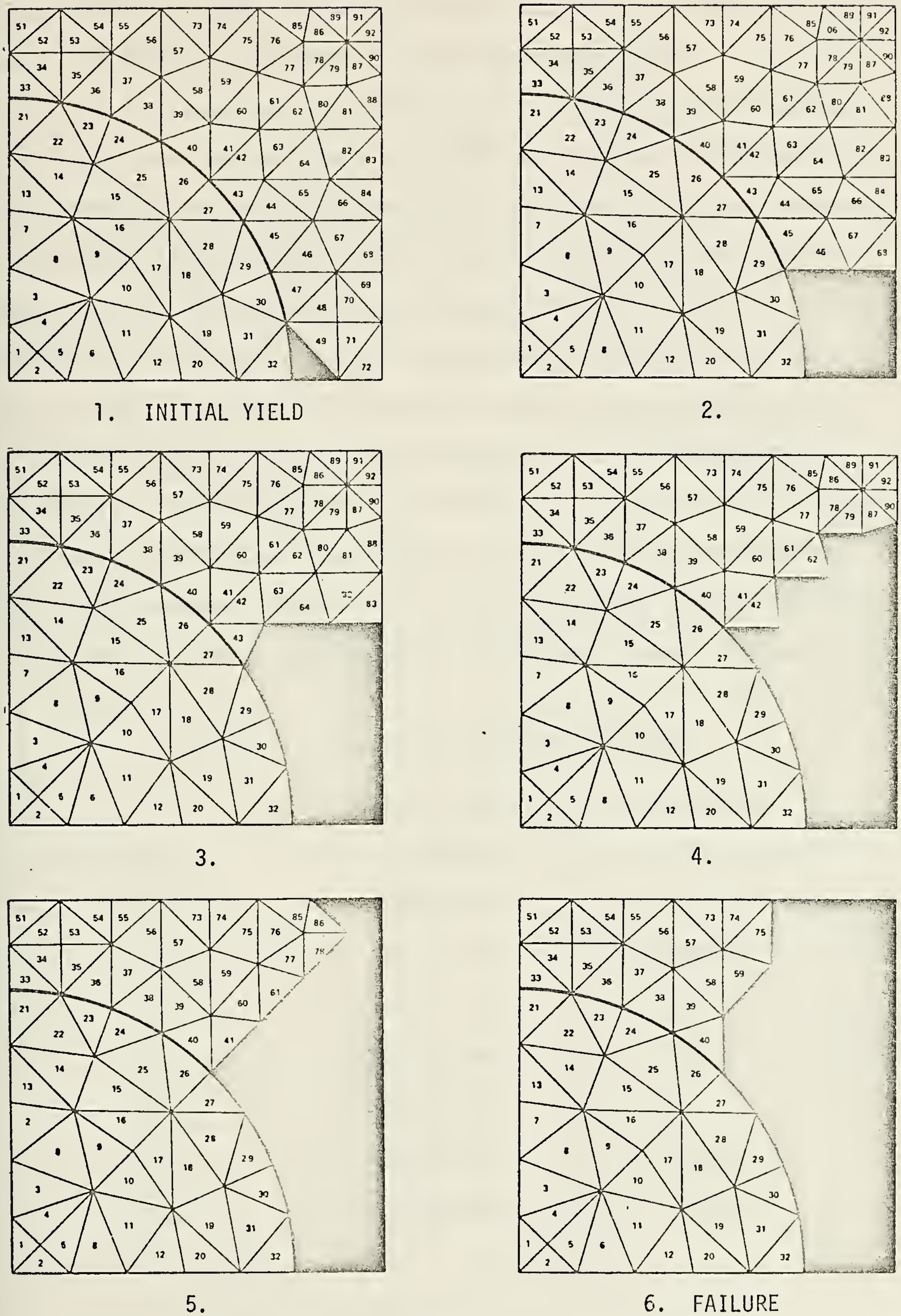


Figure 9. PROPAGATION OF PLASTIC ENCLAVE FOR BORON-ALUMINUM

elastic limit of each, the boron-aluminum composite can sustain a shear load 1300 times greater than the boron-epoxy composite.

Table 3 contains the post yield results. Continuing the loading into the plastic region, the more ductile aluminum matrix allows the boron-aluminum composites' longitudinal shear loading, S_{xz} to increase 80 percent (from 13,372 psi to 24,140 psi), while the boron-epoxy composites' S_{xz} increases 50 percent above its elastic limit (from 1044 psi to 1539 psi). These increases in longitudinal shear load correspond to a 25 percent increase in the equivalent stress of boron-aluminum (the equivalent stress in the aluminum element 50 goes from 34,600 psi to 42,500 psi), while boron-epoxy's equivalent stress increases 40 percent ($\bar{\sigma}$ increases to 4200 psi from 3000 psi). Clearly the boron-aluminum offers the designer a better shear load carrying ability than does the boron-epoxy composite when the design is allowed to reach failure.

Table 3. Results at Ultimate Strength

	Boron-aluminum	Boron-epoxy
$\sigma_{xz_{50}}$	24,444 psi	2427 psi
$\sigma_{xz_{70}}$	24,563 psi	-
$\bar{\sigma}_{max_{50}}$	42,372 psi	4212 psi
$\bar{\sigma}_{max_{70}}$	42,547 psi	-
$\bar{\sigma}_{max_{32}}$	51,097 psi	4356 psi
$\bar{\sigma}_{max_{31}}$	52,732 psi	-
S_{xz}	24,140 psi	1539 psi
S_{yz}	692 psi	33 psi

An all aluminum unit cell was subjected to the longitudinal shear loading to verify the analysis' reliability. A S_{xz} macro shear of 20,000 psi was obtained for an equivalent micro stress, $\bar{\sigma}$ of 34,600 psi at initial yield. The homogeneous cell had a uniform microstress state throughout, with $\sigma_{xz} = \bar{\sigma}/\sqrt{3} = 20{,}000$ psi and $\sigma_{yz} = 0$. The ultimate macro shear stress S_{xz} is 24,500 psi with a uniform equivalent micro stress $\bar{\sigma} = 42{,}500$ psi, resulting from $\sigma_{xz} = 24{,}000$ psi and $\sigma_{yz} = 0$ psi. We note in the case of a homogeneous cell subject to pure xz shear, there results only xz behavior, that is, in this special case post yield behavior does not give coupling between xz and yz behavior. This is evidenced by $\sigma_{yz} = 0$ psi.

The composite results were further verified by constructing an equivalent stress-strain curve (universal stress-strain curve) and then by plotting the values of equivalent stress and strain from the computer program on the same figure to show how close the analysis came to the input stress-strain curve. The elastic portion of the curve was constructed using the linear elastic relation, $\bar{\sigma} = E\bar{\varepsilon}$, while the plastic portion of the curve was constructed using the power law, $\bar{\varepsilon}^P = (\frac{\bar{\sigma}}{B})^N$. The portion of the boron-aluminum equivalent stress-strain curve where the slope becomes constant (i.e., for $\bar{\sigma} > 40{,}500$ psi) was plotted using the relation, $H' = \frac{d\bar{\sigma}}{d\bar{\varepsilon}^P}$. The power law values, N and B were calculated using the data from an original uniaxial stress-strain curve for the material. The determination of N and B for the aluminum was easily obtained, but N and B for the epoxy curve became extremely difficult to calculate due to the almost linear stress-strain curve for epoxy in the plastic region. Figures 10 and 11 show the equivalent stress versus total strain curves for the boron-epoxy and the boron-aluminum

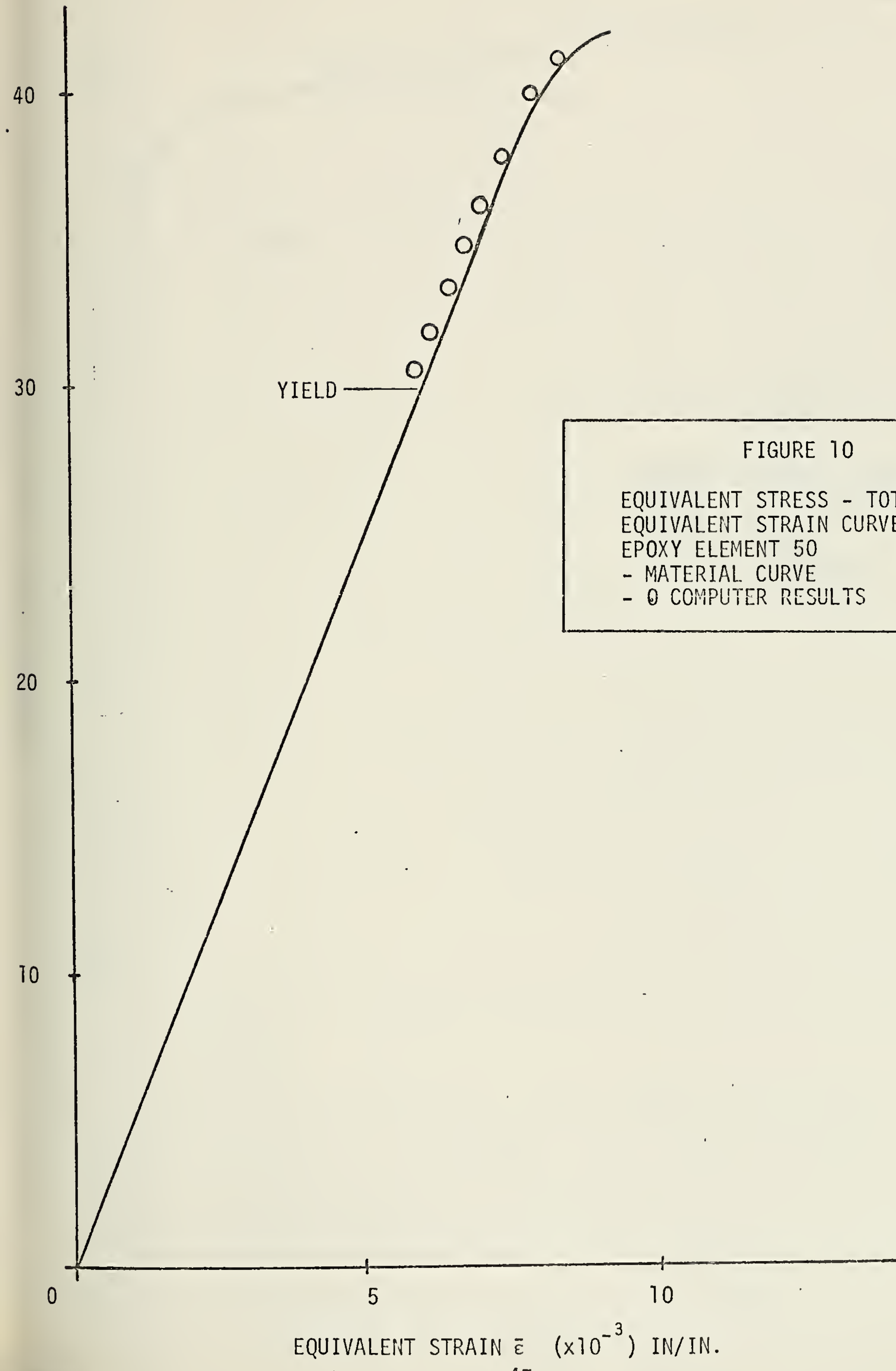


FIGURE 10

EQUIVALENT STRESS - TOTAL EQUIVALENT STRAIN CURVE FOR EPOXY ELEMENT 50
- MATERIAL CURVE
- 0 COMPUTER RESULTS

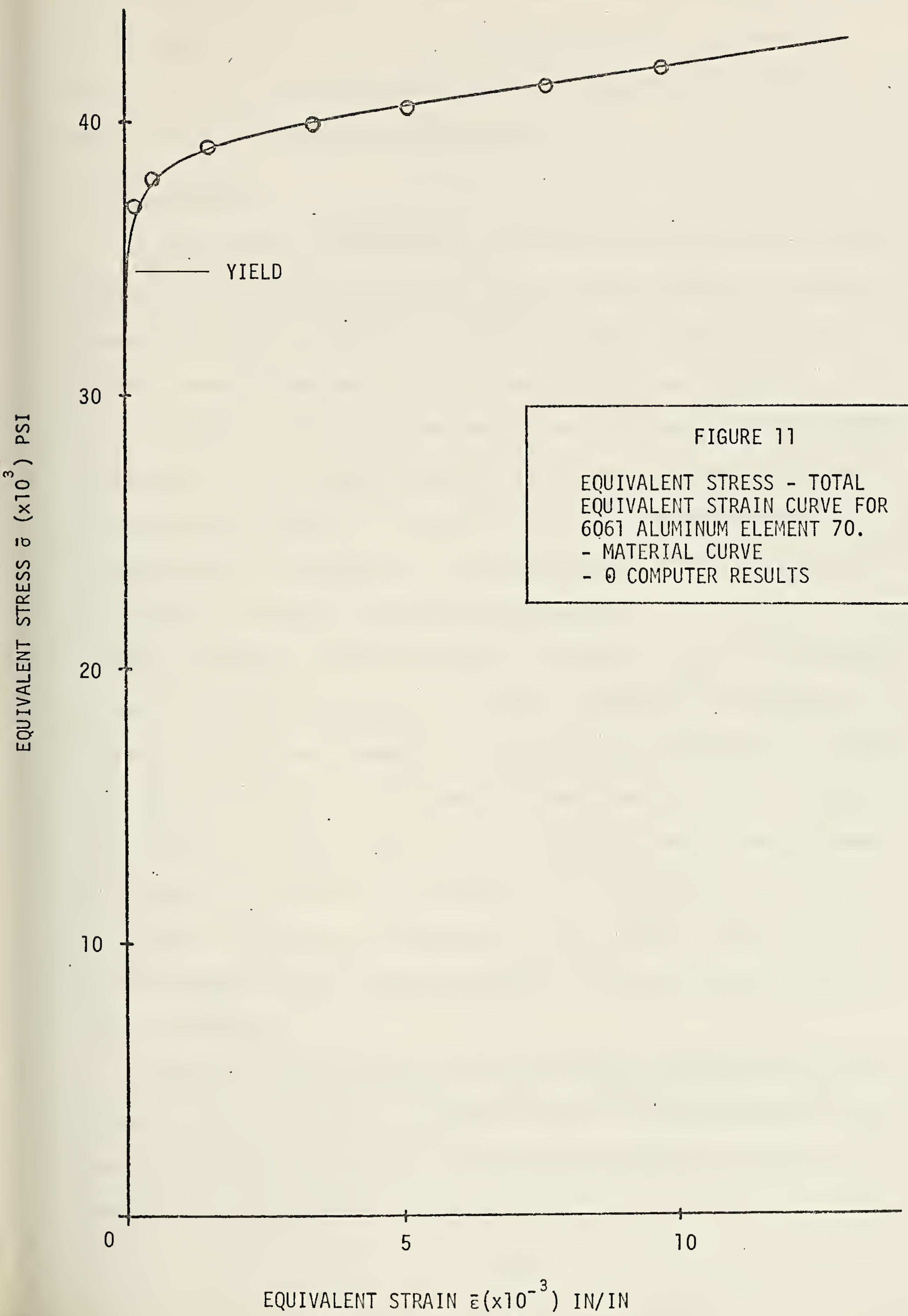


FIGURE 11

EQUIVALENT STRESS - TOTAL EQUIVALENT STRAIN CURVE FOR 6061 ALUMINUM ELEMENT 70.
- MATERIAL CURVE
- 0 COMPUTER RESULTS

respectively, while figures 12 and 13 show the equivalent stress versus the equivalent plastic strain. The close proximity of the plotted values to the curves verify small enough incremental steps were taken during the non-linear plastic analysis.

B. CONCLUSIONS

In this analysis the objective has been to develop and use a program which could determine the failure point of a unidirectional reinforced composite material subjected to longitudinal shear loading using the finite element method and to make a comparison of two composites.

The failure point was determined for the two composites and occurred at element 50 for the boron-epoxy composite and at element 70 for the boron-aluminum composite. A greater portion of the aluminum matrix yielded before failure than for the epoxy matrix. This occurrence came as a result of the more ductile aluminum matrix.

The appearance of the longitudinal macro shear S_{yz} is small compared to the longitudinal shear S_{xz}, but must be accounted for because the σ_{yz} microstresses were comparable to the σ_{xz} microstresses for the interface elements (values of 50 percent or greater were reached). Since the significant σ_{yz} stresses occur in the interface elements when the model was loaded in S_{xz} shear, xz - yz coupling may be explained as a result of the circular filament cross section. If the filament cross section had been rectangular the σ_{yz} terms and hence S_{yz} term would have been less likely to appear.

Considering all the results the boron-aluminum composite definitely seems to possess a better longitudinal shear load carrying ability than does the boron-epoxy composite. But the all aluminum cell gives the same results. Hence it would not be economically wise to use a boron-

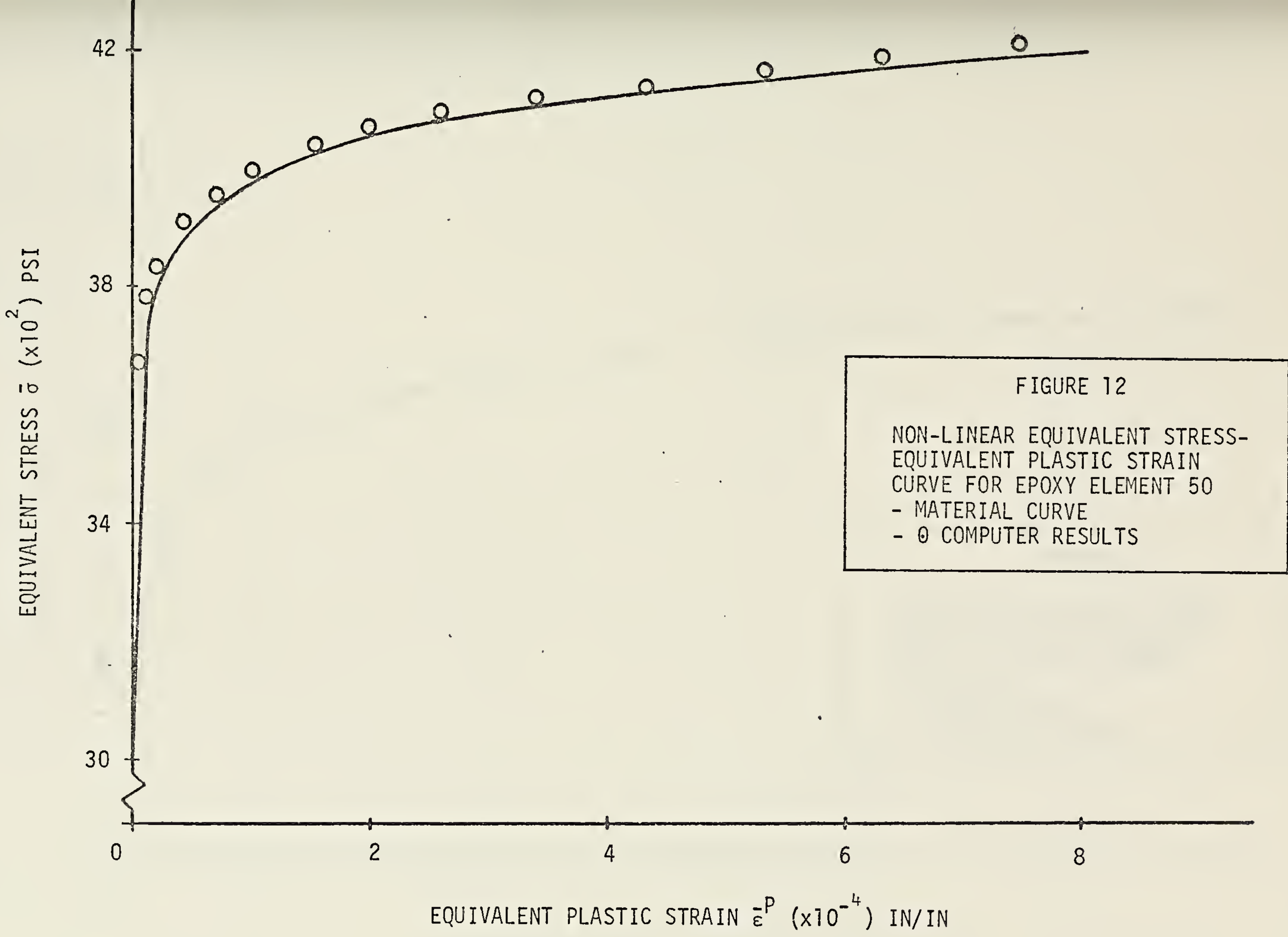


FIGURE 12

NON-LINEAR EQUIVALENT STRESS-EQUIVALENT PLASTIC STRAIN CURVE FOR EPOXY ELEMENT 50
- MATERIAL CURVE
- Θ COMPUTER RESULTS

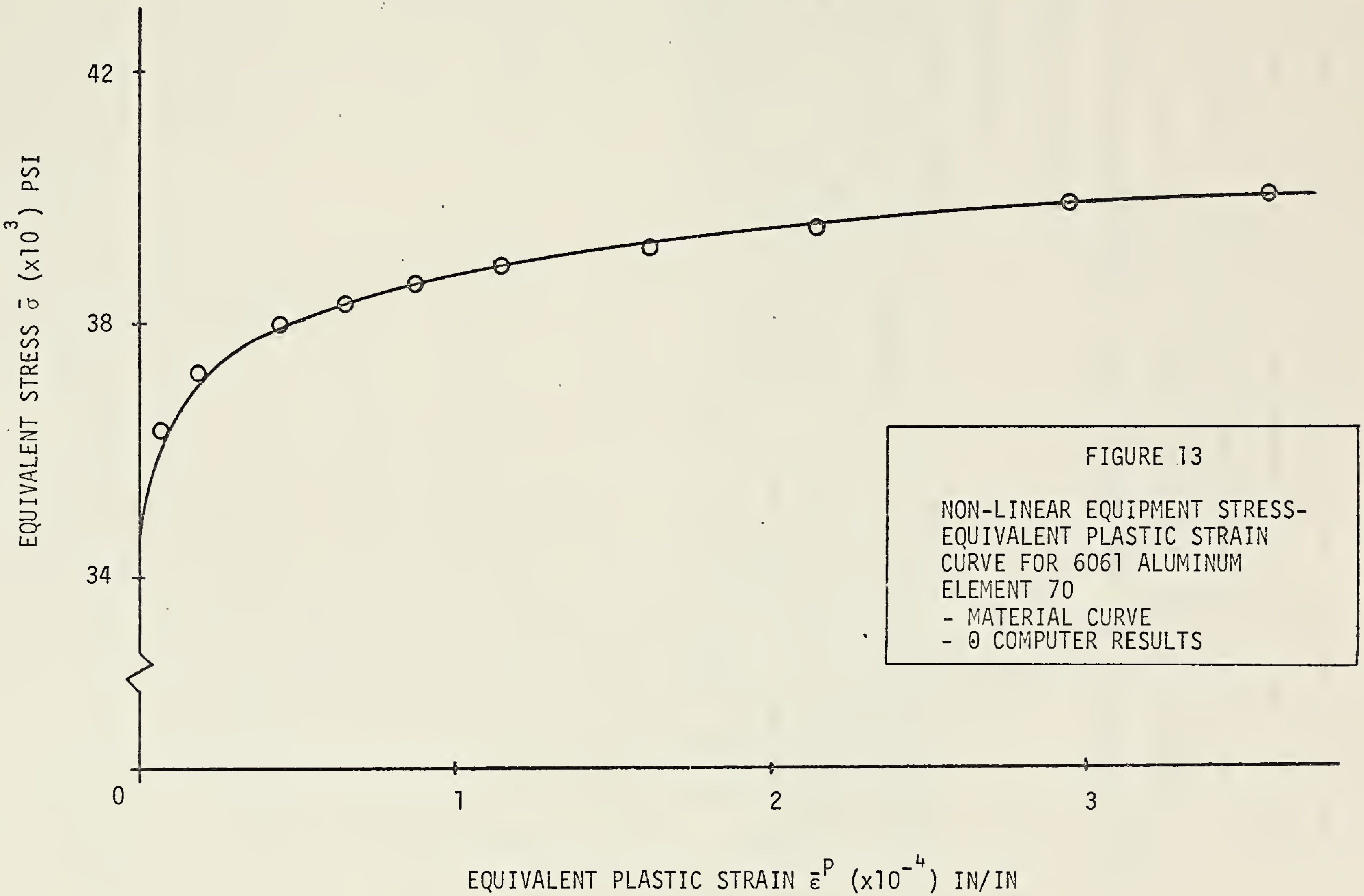


FIGURE 13

NON-LINEAR EQUIPMENT STRESS-EQUIVALENT PLASTIC STRAIN CURVE FOR 6061 ALUMINUM ELEMENT 70
- MATERIAL CURVE
- 0 COMPUTER RESULTS

aluminum laminar composite in place of a homogeneous aluminum plate if in-plane shear loading was the only load applied. The boron-aluminum composite may not fare to well when compared to an all aluminum material in the longitudinal shear mode, but from the previous analysis of Lin et al [10] we see that the all aluminum material would not approach the boron-aluminum composite when loaded in other modes (i.e., transverse and longitudinal loadings).

C. RECOMMENDATIONS

The logical extension of this analysis would be to combine it with the other loading problems studied by Lin et al [10] to construct a failure surface. Also the investigation of different shaped cross sections for the filament should be considered. These suggestions combined with a parametric analysis could lead to other interesting thesis topics.

APPENDIX A

CALCULATION OF THE STIFFNESS MATRIX USING FINITE ELEMENTS

Section three derived the stress-strain matrix for the plastic analysis and stated the stress-strain matrix for the elastic analysis. This section will show how these stress-strain matrices are expressed in finite element form so that they may be written in computer language.

In this analysis, the longitudinal shear loading problem is associated with the macroscopic stresses S_{xz} and S_{yz} and the displacement u=v=o, w=w(x,y). The finite element used in this analysis is the linear strain triangle (LST). For the longitudinal shear element the elastic strain energy is given by

$$U = \frac{1}{2}\iiint_V \sigma_{ij}\varepsilon_{ij}dV \tag{A-1}$$

which becomes for this problem

$$U = \frac{1}{2}\iiint_V (2\sigma_{xz}\varepsilon_{xz}+2\sigma_{yz}\varepsilon_{yz})dV$$

$$U = \frac{1}{2}\iiint_V (\sigma_{xz}\gamma_{xz}+\sigma_{yz}\gamma_{yz})dV \tag{A-2}$$

where σ_{xz} and σ_{yz} are the microstresses and γ_{xz} and γ_{yz} are the shear strains associated with the problem. Assuming a unit thickness

$$U = \frac{t}{2}\iint_A \langle\sigma_{xz}\sigma_{yz}\rangle\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}dA=$$

$$\frac{t}{2}\iint_A \begin{Bmatrix}\sigma_{xz}\\ \sigma_{yz}\end{Bmatrix}^T\begin{Bmatrix}\gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}dA=\frac{t}{2}\iint_A \{\sigma\}^T\{\gamma\}dA \qquad \text{(A-3}$$

The minimization of the strain energy U with respect to the displacement vecter {w} gives the stiffness matrix of an element. The derivation of the element stiffness matrix follows.

In the elastic elements,

$$\sigma_{xz} = G\gamma_{xz} \text{ and } \sigma_{yz} = G\gamma_{yz}$$

$$\begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \{\sigma\} = [D^E]\ \{\gamma\} \qquad \text{(A-4)}$$

Substituting into equation (A-3)

$$U = \frac{t}{2}\iint_A \{\gamma\}^T[D^E]\{\gamma\}dA \qquad \text{(A-5)}$$

Before the derivation may continue the array {γ} must be defined.

$$\{\gamma\} = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix} \begin{Bmatrix} \gamma_{xz_1} \\ \gamma_{xz_2} \\ \gamma_{xz_3} \\ \hline \gamma_{yz_1} \\ \gamma_{yz_2} \\ \gamma_{yz_3} \end{Bmatrix}$$

$$= \underset{2 \times 6}{[\ \zeta\ ]} \begin{Bmatrix} \gamma_{xz_i} \\ \hline \gamma_{yz_i} \end{Bmatrix} \qquad i = 1, 2, 3 \qquad \text{(A-6)}$$

where γ_i are triangular coordinates.

The γ_{xz}'s and γ_{yz}'s go from one to three for the corner nodal points of a linear strain triangle. Now the nodal shear strains may be expressed in terms of the nodal displacements

$$\{\gamma_{xz_i}\} = \frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \left\langle N_1 N_2 N_3 N_4 N_5 N_6 \right\rangle \Big|_i \begin{Bmatrix} w_1 \\ \vdots \\ w_6 \end{Bmatrix}$$

$$= \left\langle N_{x_1} N_{x_2} N_{x_3} N_{x_4} N_{x_5} N_{x_6} \right\rangle \Big|_i \begin{Bmatrix} w_1 \\ \vdots \\ w_6 \end{Bmatrix} \qquad \text{(A-7)}$$

and

$$\{\gamma_{yz_i}\} = \frac{\partial w}{\partial y} = \frac{\partial}{\partial y} \left\langle N_1 N_2 N_3 N_4 N_5 N_6 \right\rangle \Big|_i \begin{Bmatrix} w_1 \\ \vdots \\ w_6 \end{Bmatrix}$$

$$= \left\langle N_{y_1} N_{y_2} N_{y_3} N_{y_4} N_{y_5} N_{y_6} \right\rangle \Big|_i \begin{Bmatrix} w_1 \\ \vdots \\ w_6 \end{Bmatrix} \qquad \text{(A-8)}$$

where the N_{x_i} and N_{y_i} are the first partial of the shape functions with respect to x and y respectively. Evaluating equations (A-7) and (A-8) at each of the six nodes,

$$\gamma_{xz_{1\ldots6}} = \left\langle N_{x_1} \ldots N_{x_6} \right\rangle \Big|_{1\ldots6} \{w\} \text{ and} \qquad \text{(A-9)}$$

$$\gamma_{yz_{1\ldots6}} = \left\langle N_{y_1} \ldots N_{y_6} \right\rangle \Big|_{1\ldots6} \{w\} \qquad \text{(A-10)}$$

Combining results and putting equations (A-9) and (A-10) in matrix form

$$\underset{6X1}{\{\gamma_{xz}\}} = \underset{6X6}{[zx]} \ \underset{6X1}{\{w\}} \qquad \text{(A-11)}$$

$$\underset{6X1}{\{\gamma_{yz}\}} = \underset{6X6}{[zy]} \ \underset{6X1}{\{w\}} \qquad \text{(A-12)}$$

Substituting equations (A-11) and A-12) into (A-6) we have

$$\{\gamma\} = [\xi] \begin{bmatrix} [zx] \\ [zy] \end{bmatrix} \{w\} \qquad \text{(A-13)}$$

$$= [\xi]\ [z]\ \{w\}$$

Now using the equation (A-13) and equation (A-5) we finally have

$$U = \frac{t}{2} \iint_A \{w\}^T [z]^T [\xi]^T [D^E]^T [\xi][z]\{w\} dA \qquad \text{(A-14)}$$

Letting $[B] = [\xi][z]$ we have

$$U = \frac{t}{2} \iint_A \{w\}^T [B]^T [D^E]^T [B] \{w\} dA \qquad \text{(A-15)}$$

Minimizing U with respect to {w} and integrating

$$[k^E] = [B]^T [D^E][B] tA \qquad \text{(A-16)}$$

which is identical to equation (4-11)

For the plastic elements the stiffness matrix becomes more involved. The stress-strain relatives are a function of the loading (see equation (3-26)). The only difference is that $[D^P]$ takes the place of $[D^E]$ in the plastic stiffness matrix, thus

$$[k^P] = [B]^T [D^P][B] tA. \qquad \text{(A-17)}$$

APPENDIX B

SOME DERIVATIONS OF EQUATIONS

In this appendix will be presented some important derivations necessary to the understanding of the elastic-plastic analysis. There are two proofs of the scale factor R used in section four and five. The first proof is geometrical in nature, while the second is an algebraic derivation. The derivation of the gamma, which is necessary for the scale factor R derivation is also included. Finally it is shown that the constant C of equation (2-18) equals 3/2.

A. GEOMETRIC DERIVATION OF EQUATION (4-15)

Here we obtain equation (4-15) through a geometric viewpoint. The basis of the discussion is contained in figure 14. In figure 14 $\bar{\sigma}$ is the present equivalent stress of the elastic element and $\Delta\bar{\sigma}^T$ denotes the increment of $\bar{\sigma}$ induced by the displacement increment $\{\Delta w^T\}$. $\bar{\sigma}$ and $\Delta\bar{\sigma}^T$ are represented respectively by $\overline{OP}$ and $\overline{PS}$, while $\overline{\Delta\sigma}^T = \overline{PR}$. For the longitudinal shear problem $(\bar{\sigma}+\overline{\Delta\sigma}^T)^2$ (see equation (B-7)) and $(\overline{\Delta\sigma}^T)^2$ (see equation (B-13)) determines the load increment sufficient to just cause yield in each elastic element.

$$PR = \overline{\Delta\sigma}^T \qquad R = \frac{\overline{PQ}}{\overline{PR}}$$

Using the cosine law,

$$\text{Let } x = \frac{\bar{\sigma}^2+(\Delta\bar{\sigma}^T)^2 - (\bar{\sigma}+\Delta\bar{\sigma}^T)^2}{2\bar{\sigma}\,\Delta\bar{\sigma}^T}$$

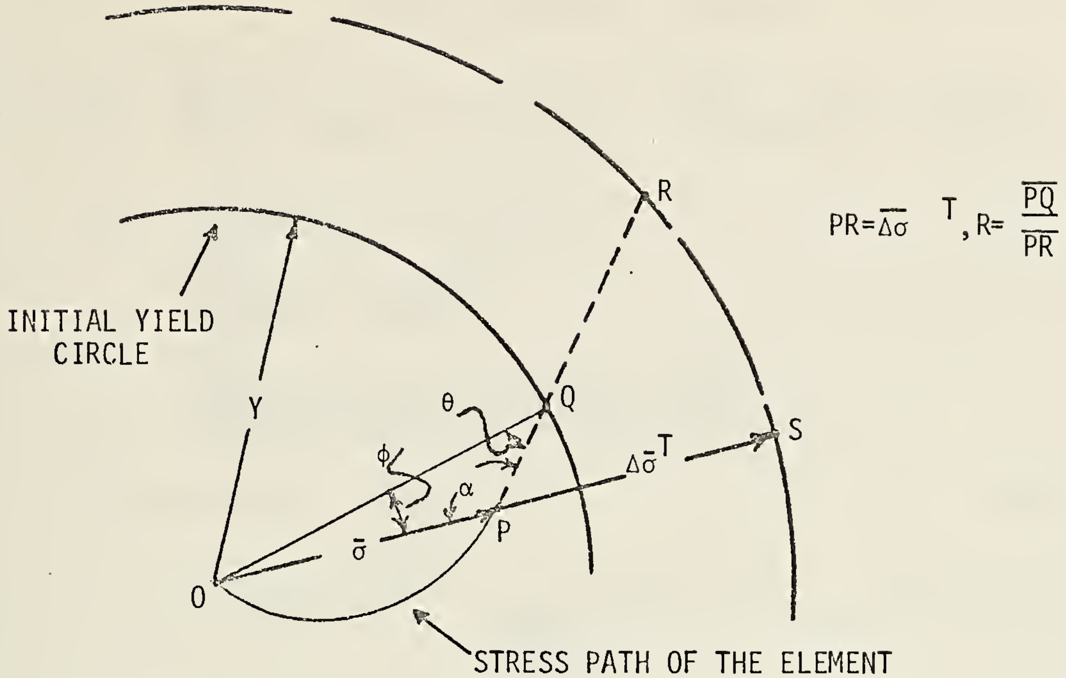


FIGURE 14. DETERMINATION OF THE LOAD INCREMENT TO JUST YIELD THE ELASTIC ELEMENT.

and

$$\mathrm{COS}\alpha = x \qquad \alpha = \mathrm{ARCCOS}x \qquad \mathrm{SIN}\alpha = \sqrt{1-x^2}$$

Using the sine law,

$$\frac{\overline{QO}}{\mathrm{SIN}\alpha} = \frac{\overline{PQ}}{\mathrm{SIN}\phi} = \frac{\overline{OP}}{\mathrm{SIN}\Theta} \qquad \phi = 180 - (\alpha+\Theta)$$

$$\overline{QO} = Y \qquad \overline{OP} = \bar{\sigma}$$

$$\frac{Y}{\sqrt{1-x^2}} = \frac{\overline{PQ}}{\mathrm{SIN}\phi} \qquad \frac{Y}{\sqrt{1-x^2}} = \frac{\bar{\sigma}}{\mathrm{SIN}\Theta}$$

$$\mathrm{SIN}\Theta = \frac{\bar{\sigma}\,(\sqrt{1-x^2})}{Y} = z$$

$$\Theta = \mathrm{ARCSIN}Z$$

$$\phi = 180 - (\text{ARCCOSX} + \text{ARCSINZ})$$

$$\frac{Y}{\sqrt{1-x^2}} = \frac{\overline{PQ}}{\text{SIN}(180-(\text{ARCCOSX} + \text{ARCSINZ}))} = \frac{\overline{PQ}}{\text{SIN}(\text{ARCCOSX} + \text{ARCSINZ})}$$

LET β = ARCCOSX and γ = ARCSINZ

$$= \frac{\overline{PQ}}{\text{SIN}\beta\text{COS}\gamma + \text{COS}\beta\text{SIN}\gamma}$$

$$= \frac{\overline{PQ}}{[(\sqrt{1-x^2})(\sqrt{1-z^2}) + (x)(z)]} = \frac{Y}{\sqrt{1-x^2}}$$

$$\overline{PQ} = Y\{\sqrt{1-z^2} + \frac{xz}{1-x^2}\} \qquad \text{(B-1)}$$

$$= Y\sqrt{1-(\tfrac{\bar{\sigma}}{Y})^2(1-x^2)} + \frac{x(\tfrac{\bar{\sigma}}{Y})\sqrt{(1-x^2)}Y}{\sqrt{1-x^2}}$$

$$= \sqrt{Y^2 - \bar{\sigma}^2(1-x^2)} + x\bar{\sigma}$$

$$\overline{PQ} = \{[(Y^2-\bar{\sigma}^2)4(\overline{\Delta\sigma}^T)^2+x^2]/4(\Delta\bar{\sigma}^T)^2\}^{\frac{1}{2}} +$$

$$(\bar{\sigma}^2)+(\Delta\bar{\sigma}^T)^2-(\bar{\sigma}+\Delta\bar{\sigma}^T)^2)/2(\overline{\Delta\sigma}^T) \qquad \text{(B-2)}$$

Expanding terms and letting

$$\Gamma = (\overline{\Delta\sigma}^T)^2-2\bar{\sigma}\Delta\bar{\sigma}^T - (\Delta\bar{\sigma}^T)^2 \qquad \text{(B-3)}$$

$\overline{PQ}$ becomes

$$\overline{PQ} = [\{(Y^2-\bar{\sigma}^2)4(\overline{\Delta\sigma}^T)^2+\Gamma^2\}^{\frac{1}{2}}+\Gamma]/2\overline{\Delta\sigma}^T \qquad \text{(B-4)}$$

$$R = \frac{\overline{PQ}}{\overline{PR}} = \frac{[\{(Y^2-\bar{\sigma}^2)4(\overline{\Delta\sigma}^T)^2+\Gamma^2\}^{\frac{1}{2}}+\Gamma)]}{2(\overline{\Delta\sigma}^T)^2} \qquad \text{(B-5)}$$

B. ALGEBRAIC DERIVATION OF EQUATION (4-15)

The next derivation of R may be expressed in general terms but is instead derived with the particular problem of longitudinal shear loading in mind.

When $\bar{\sigma}+\Delta\bar{\sigma} = Y$ yield occurs, $\Delta\bar{\sigma}_T$ is produced by the incremental test displacement Δw^T.

$$\bar{\sigma}^2 = 3[\sigma_{xz}{}^2+\sigma_{yz}{}^2] \tag{B-6}$$

$$(\bar{\sigma}+\Delta\bar{\sigma}^T)^2 = 3[(\sigma_{xz}+\Delta\sigma_{xz}{}^T)^2+(\sigma_{yz}+\Delta\sigma_{yz}{}^T)^2] \tag{B-7}$$

$$\bar{\sigma}^2+2\bar{\sigma}\ \Delta\bar{\sigma}^T+(\Delta\bar{\sigma}^T)^2 = 3[\sigma_{xz}{}^2+2\sigma_{xz}\Delta\sigma_{xz}{}^T$$

$$+ (\Delta\sigma_{xz}{}^T)^2+\sigma_{yz}{}^2+2\sigma_{yz}\ \Delta\sigma_{yz}{}^T+(\Delta\sigma_{yz}{}^T)^2]$$

Also when $(\sigma+R\Delta\bar{\sigma}^T)^2 = Y^2$ element is at yield:

$$Y^2 = 3[\sigma_{xz}{}^2+2R\sigma_{xz}\ \Delta\bar{\sigma}_{xz}{}^T+R^2(\Delta\bar{\sigma}_{xz}{}^T)^2$$

$$+ \sigma_{yz}{}^2+2R\sigma_{yz}\ \Delta\bar{\sigma}_{yz}{}^T+R^2(\Delta\bar{\sigma}_{yz}{}^T)^2] \tag{B-8}$$

$$Y^2 = \bar{\sigma}^2+3[2R(\sigma_{xz}\Delta\bar{\sigma}_{xz}{}^T+\sigma_{yz}\ \Delta\bar{\sigma}_{yz}{}^T)$$

$$+ R^2(\Delta\bar{\sigma}_{xz}{}^{T2} +\Delta\sigma_{yz}{}^{T2})]$$

$$\Gamma = -6[\sigma_{xz}\Delta\bar{\sigma}_{xz}{}^T+\sigma_{yz}\ \Delta\bar{\sigma}_{yz}{}^T] \tag{B-9}$$

$$(\overline{\Delta\sigma}^T)^2 = 3[(\Delta\bar{\sigma}_{xz}{}^T)^2+(\Delta\bar{\sigma}_{yz}{}^T)^2] \tag{B-10}$$

Making the necessary substitutions,

$$(\overline{\Delta\sigma}^T)^2R^2-\Gamma R+\bar{\sigma}^2-Y^2 = 0$$

$$R = \frac{\Gamma+\sqrt{\Gamma^2+4(\overline{\Delta\sigma}^T)^2(Y^2-\bar{\sigma}^2)}}{2(\overline{\Delta\sigma}^T)^2} \tag{B-11}$$

C. DERIVATION OF EQUATION (4-16) FOR Γ

From the original postulate

$$\Gamma = (\overline{\Delta\sigma}^T)^2-2\bar{\sigma}\Delta\bar{\sigma}^T-(\Delta\bar{\sigma}^T)^2 \tag{B-12}$$

$$\overline{\Delta\sigma}^T = (\tfrac{3}{2}\Delta\sigma_{ij}^{\prime T}\ \Delta\sigma_{ij}^{\prime T})^{\frac{1}{2}}$$

which equals $\overline{PR}$ in figure 14. The term $\overline{\Delta\sigma}^T$ becomes upon expansion

$$\overline{\Delta\sigma}^T = \{\tfrac{3}{2}(\Delta\sigma_{xx}^{\prime T2}+\Delta\sigma_{yy}^{\prime T2}+\Delta\sigma_{zz}^{\prime T2}+2((\Delta\sigma_{xy}^{\prime T})^2+(\Delta\sigma_{yz}^{\prime T})^2$$

$$+ (\Delta\sigma_{zx}^{\prime T})^2))\}^{\frac{1}{2}} \tag{B-13}$$

The equivalent stress vector is

$$\tfrac{2}{3}\bar{\sigma}^2 = \sigma_{xx}^{\prime 2}+\sigma_{yy}^{\prime 2}+\sigma_{zz}^{\prime 2}+2(\sigma_{xy}^{\prime 2}+\sigma_{yz}^{\prime 2}+\sigma_{xz}^{\prime 2})$$

$$(\bar{\sigma}+\Delta\bar{\sigma}^T)^2 = \bar{\sigma}^2+2\bar{\sigma}\ \Delta\bar{\sigma}^T+(\Delta\bar{\sigma}^T)^2 \tag{B-14}$$

putting $\bar{\sigma} + \Delta\bar{\sigma}^T$ in the form of the equivalent stress vector

$$\tfrac{2}{3}(\bar{\sigma}+\Delta\bar{\sigma}^T)^2 = (\bar{\sigma}_{xx}+\Delta\bar{\sigma}_{xx}^T)^{\prime 2}+(\bar{\sigma}_{yy}+\Delta\bar{\sigma}_{yy}^T)^{\prime 2}+$$

$$(\bar{\sigma}_{zz}+\Delta\bar{\sigma}_{zz}^T)^{\prime 2}+2[(\bar{\sigma}_{xy}+\Delta\bar{\sigma}_{xy}^T)^{\prime 2}+(\bar{\sigma}_{yz}+\Delta\bar{\sigma}_{yz}^T)^{\prime 2}$$

$$+ (\bar{\sigma}_{zx}+\Delta\bar{\sigma}_{zx}^T)^{\prime 2}]$$

$$\tfrac{4}{3}\bar{\sigma}\Delta\bar{\sigma}^T+\tfrac{2}{3}(\Delta\bar{\sigma}^T)^2 = \tfrac{2}{3}(\bar{\sigma}+\Delta\bar{\sigma}^T)^2-\tfrac{2}{3}\bar{\sigma}^2 \tag{B-15}$$

Now the outcome is subtracted from $(\overline{\Delta\sigma}^T)^2$ to give Γ. For this analysis Γ becomes

$$\Gamma = (\overline{\Delta\sigma}^T)^2-2\bar{\sigma}\Delta\bar{\sigma}^T-(\Delta\bar{\sigma}^T)^2$$

$$= -6[\bar{\sigma}_{xy}\ \overline{\Delta\sigma}_{xy}^T+\bar{\sigma}_{yz}\ \overline{\Delta\sigma}_{yz}^T] \tag{B-16}$$

D. DERIVATION OF C IN EQUATION (2-18)

The proof that constant "C" of equation (2-18) is simple and straightforward. Equation (2-19) for a tension test and principal strains becomes

$$\int \overline{d\varepsilon^P} = \int \sqrt{\tfrac{2}{3}(d\varepsilon_{ij}^P\, d\varepsilon_{ij}^P)}^{\frac{1}{2}} =$$

$$\sqrt{\tfrac{2}{3}}\int\{(d\varepsilon_1^P)^2+(-\frac{d\varepsilon_1^P}{2})^2+(-\frac{d\varepsilon_1^P}{2})^2\}^{\frac{1}{2}} = \int d\varepsilon_1^P \qquad \text{(B-17)}$$

and from equation (3-11) the equivalent stress becomes

$$\bar{\sigma} = \sqrt{\tfrac{3}{2}}(\sigma_{ij}'\,\sigma_{ij}')^{\frac{1}{2}} = \sqrt{\tfrac{3}{2}}\{\sigma_1'^2+\sigma_2'^2+\sigma_3'^2\}^{\frac{1}{2}} \qquad \text{(B-18)}$$

where σ_1', σ_2' and σ_3' are the principal deviatoric stresses. Equation (B-18) simplifies to

$$\bar{\sigma} = \sqrt{\tfrac{3}{2}}\,\{(\tfrac{2}{3}\sigma_1)^2+(-\tfrac{1}{3}\sigma_1)^2+(-\tfrac{1}{3}\sigma_1)^2\}^{\frac{1}{2}} = \sigma_1$$

and for tension $\sigma_1' = \frac{2}{3}\bar{\sigma}$. Now substituting into equation (2-17)

$$d\lambda = \frac{d\varepsilon_1^P}{\sigma_1^1} = \frac{\overline{d\varepsilon^P}}{\frac{2}{3}\bar{\sigma}} = \frac{3}{2}\,\frac{\overline{d\varepsilon^P}}{\bar{\sigma}}$$

APPENDIX C

THE COMPUTER PROGRAM

The computer program performs an elastic-plastic analysis for a unidirectional reinforced composite material subjected to longitudinal shear loading. The main sections of the program are listed and described below.

Main - Constructs the elastic stiffness matrix and modifies it according to loading and boundary conditions. It calls the equation solver and then scales the elements in the elastic and plastic analysis and controls further calling sequences.

Post - Constructs the plastic stiffness matrices for the elements in the plastic range.

Caltau - Calculates element strains and stresses; $\varepsilon = Bw$, $\sigma = D^E \varepsilon$ and $\sigma = D^P \varepsilon$.

Bandec - Solves banded matrix problems ($Kw=F$, or $K(\Delta w)=\Delta F$).

MULI - Multiplies a symmetric banded matrix with a vector.

This portion of the discussion of the computer program is devoted to assumptions made in writing the program and are listed here for convenience.

1. The properties of the matrix material may be different in compression and in tension, YMC and YMT respectively. This assumption explains the dilatation variable SII's inclusion in the program. SII was the variable that determined if a matrix element was in compression or tension. It was included to preserve the generality of the program but does not enter the analysis in the particular case of longitudinal shear loading.

2. If an element is within a particular fraction (called TEST) of the yield value, it is included as a yielded element. In this analysis, TEST was chosen as 0.995. Therefore for calculations any element which is at .995 times the yield value is at yield (TEST = .995). After an increment it can be determined whether an element which was close to yield and assumed to yield, actually did. In other words, the size of the incremental step and TEST should form a consistent pair.

3. For purposes of scaling the load to take the critical element to yield, the scale factor R_{min} is restricted to be less than some maximum (RYMIN). This is imposed to limit the size of an incremental step taken along the stress-strain curve. If too large size steps are taken, the elements in the plastic region will provide inaccurate results (see figure 2). Also at a certain point in the stress-strain curve larger R_{min} values will be possible which would reduce the calculation time. These maximum values for R_{min} along the stress-strain curve must be altered according to the materials being used. They are also a result of experience in working with a certain material to produce better results.

4. Also in scaling the load to take the critical element to yield the filament elements will be ignored. The yield of a filament is much higher than the yield value for a matrix element.

5. For purposes of computer efficiency the elements of the stiffness matrix were derived to facilitate their use in the construction of the stiffness matrix in the main program and the strain column matrix in the subroutine Caltau. Therefore the zx and zy variable arrays listed are the first partial of the shape functions with respect to x and y respectively.

$$[zx]_{3\times6} = \begin{bmatrix} \left\langle \frac{\partial N_i}{\partial x} \right\rangle \text{at } 1 \\ \left\langle \frac{\partial N_i}{\partial x} \right\rangle \text{at } 2 \\ \left\langle \frac{\partial N_i}{\partial x} \right\rangle \text{at } 3 \end{bmatrix} \qquad [zy]_{3\times6} = \begin{bmatrix} \left\langle \frac{\partial N_i}{\partial y} \right\rangle \text{at } 1 \\ \left\langle \frac{\partial N_i}{\partial y} \right\rangle \text{at } 2 \\ \left\langle \frac{\partial N_i}{\partial y} \right\rangle \text{at } 3 \end{bmatrix} \qquad i = 1, \ldots, 6.$$

6. For $H' = d\bar{\sigma}/d\overline{\varepsilon^P}$ a two point curve fitting scheme is used to approximate the plastic portion of the stress-strain curve. For example assuming the plastic strain-stress relation can be expressed as the power low $\varepsilon^P = (\frac{\bar{\sigma}}{B})^N$

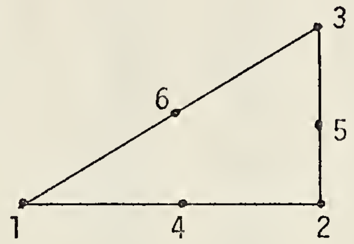
$$\frac{d\bar{\sigma}}{d\varepsilon^P} = B\left(\frac{B}{\bar{\sigma}}\right)^{N-1}$$

The values of B and N may vary for different material and for tension and compression of the matrix. The calculation of N and B results from solving the equation $\bar{\varepsilon}^P = (\frac{\bar{\sigma}}{B})^N$ for two points on the plastic portion of the curve. The selection of the two points is important and should approximately divide the non-linear portion equally. The points are chosen in this manner so that large errors do not result. This method is an accurate and simple procedure.

Other curve fitting expressions giving more accurate results may be used. It is felt that more accurate results are not necessary and do not yield that significant of an improvement in the results for the amount of time that would have to be expended.

7. Data input is another important aspect of programming. The only possible trouble area exists in numbering the nodes of a triangular

element. The usual convention is followed by numbering the corner nodes first and then the intermediate nodes in a counter-clockwise fashion. For example, the numbering sequence should appear as:



```
C
C
C
C
C     ****************************************************************
C     *                                                              *
C     *                                                              *
C     *                                                              *
C     *  THIS PROGRAM DOES THE ELASTIC-PLASTIC ANALYSIS OF UNI-      *
C     *  DIRECTIONAL REINFORCED FILAMENT COMPOSITES UNDER LONGITUDINAL *
C     *  SHEAR LOADING.                                              *
C     *                                                              *
C     *  THE PROGRAM EMPLOYS LINEAR STRAIN TRIANGLES IN THE FINITE   *
C     *  ELEMENT ANALYSIS, THE PRANDTL-REUSS EQUATIONS, AND THE VON  *
C     *  MISES YIELD CRITERION.                                      *
C     *                                                              *
C     *  THE USUAL CASE OF STIFF FILAMENT AND SOFT MATRIX AS WELL AS THE *
C     *  SPECIAL CASES CAN BE ANALYZED.                              *
C     *                                                              *
C     *    1.)  FOR THE HOMOGENEOUS MATERIAL SET NFLEL EQUAL TO ONE AND *
C     *    SET THE PROPERTIES OF THE FILAMENT ELEMENTS EQUAL TO THE  *
C     *    PROPERTIES OF THE MATRIX.                                 *
C     *                                                              *
C     *    2.)  FOR THE SOFT FILAMENT, STIFF MATRIX CASE ( I.E. YOUNG'S *
C     *    MODULUS OF THE MATRIX GREATER THAN YOUNG'S MODULUS OF THE *
C     *    FILAMENT); THIS CASE IS AUTOMATICALLY TAKEN CARE OF IN THE *
C     *    PROGRAM WITH A MESSAGE.                                   *
C     *                                                              *
C     *                                                              *
C     *                                                              *
C     ****************************************************************
C
C
C
C
      COMMON/XYZ/A(92),SXZ(92),SYZ(92),ZX(92,3,6),ZY(92,3,6),CP(92,2,2),
     1ITEM(92,6)
      COMMON/XY/EQT(92),GRANK(215,51),GRAND(215,51),HPR(92),T(92),SII(92
     1),ZETA(3,3),NPEL(92)
      COMMON/XZ/UE(215)
C
      DIMENSION X(2,215),Y(2,62 ),ZK(6,6),P(215),PV(215),UBC(215),
     1UVC(215),DELTXZ(92),DELTYZ(92),DELSXZ(92),DELSYZ(92),DELEXZ(92),
     2DELEYZ(92),DELEQT(92),DELEP(92)
      DIMENSION EXZ(92),EYZ(92),TAUXZ(92),TAUYZ(92),G(92),TOPLSN(92),
     1DLPEXZ(92),DLPEYZ(92)
      DIMENSION NCP(62),NOLO(215),NUDISP(215),NBDY(215),IELBX(20),
     1IELBY(20)
```

```
C                                                                        MA0490
C                                                                        MA0500
C                                                                        MA0510
C     ******************************************************************** MA0520
C                                                                        MA0530
C                                                                        MA0540
      READ(5,9999) TITLE                                                 MA0550
    1 READ(5,10) NUMEL,NUMNP,NBAN,NUMDF,NAL,NBC,NUBC,NFLEL               MA0560
   10 FORMAT(8I10)                                                       MA0570
      WRITE(6,11) NUMEL,NUMNP,NBAN,NUMDF,NAL,NBC,NUBC,NFLEL              MA0580
   11 FORMAT(1H12X,'NUMBER OF ELEMENTS =',I5//2X,'NUMBER OF NODAL POINTS MA0590
     1=',I5//2X,'BANDWIDTH OF GRANK=',I5//2X,         'NUMBER OF DEGR    MA0600
     2EES OF FREEDOM=',I5//2X,'NUMBER OF APPLIED LOADS =',I5//2X,'NUMBER MA0610
     3 CF BOUNDARY CONDITIONS =',I5//2X,'NUMBER CF NON ZERC DISPLACEMENT MA0620
     4 BC=',I5//2X,'NUMBER OF FILAMENT ELEMENTS=',I5)                    MA0630
C                                                                        MA0640
      READ(5,15) NDFNP,NTCP,NELBX,NELBY,NINC,MODEL                       MA0650
   15 FORMAT(6I10)                                                       MA0660
      WRITE(6,16) NDFNP,NTCP,NELBX,NELBY,NINC,MCDEL                      MA0670
   16 FORMAT(//2X,'NUMBER OF DOF AT EACH NP=',I5//2X,'NUMBER OF TRIANGLE MA0680
     1 CORNER POINTS=',I5//2X,'NUMBER OF ELEMENTS ON THE X FACE=',I5//2X MA0690
     2,'NUMBER OF ELEMENTS ON THE Y FACE=',I5//2X,'MAXIMUM NUMBER OF PLA MA0700
     3STIC STEPS',I5//2X,'MODEL ',I5)                                    MA0710
C                                                                        MA0720
      READ(5,10) (IELBX(I),I=1,NELBX)                                    MA0730
      READ(5,10) (IELBY(I),I=1,NELBY)                                    MA0740
      WRITE(6,17)                                                        MA0750
   17 FORMAT(////2X,'ELEMENTS',5X,'ON THE X FACE',5X,'ON THE Y FACE')    MA0760
      WRITE(6,18) (I,IELBX(I),IELBY(I),I=1,NELBX)                        MA0770
   18 FORMAT(1X,I5,9X,I5,15X,I5)                                         MA0780
C                                                                        MA0790
      READ(5,20) ((Y(I,J),I=1,2),J=1,NTCP)                               MA0800
   20 FORMAT(2F10.5)                                                     MA0810
      WRITE(6,21) MODEL                                                  MA0820
   21 FORMAT(//2X,'GEOMETRY OF MODEL',I5)                                MA0830
      WRITE(6,22)                                                        MA0840
   22 FORMAT(//2X,'CORNER',12X,' X COORDINATE',8X,'Y COORDINATE'/)       MA0850
      WRITE(6,23) (J,Y(1,J),Y(2,J),J=1,NTCP)                             MA0860
   23 FORMAT(3X,I5,12X,F10.5,10X,F10.5)                                  MA0870
C                                                                        MA0880
      READ(5,30) ((ITEM(I,J),J=1,6),T(I),I=1,NUMEL)                      MA0890
   30 FORMAT(6I10,F10.5)                                                 MA0900
      WRITE(6,31)                                                        MA0910
   31 FORMAT(//4X,'ELEMENT',5X,'NP 1',10X,'NP 2',10X,'NP 3',10X,'NP 4',  MA0920
     110X,'NP 5',10X,'NP 6',10X,'THICKNESS'//)                           MA0930
      WRITE(6,32) (J,(ITEM(J,K),K=1,6),T(J),J=1,NUMEL)                   MA0940
   32 FORMAT(3X,I5,I11,5I14,F20.5)                                       MA0950
C                                                                        MA0960
```

```
      READ(5,40) POISF,POISM,EF,EM,YF,YMT,YMC,EQTMAX
   40 FORMAT(2F10.5,6E10.4)
      WRITE(6,41) POISF,POISM,EF,EM,YF,YMT,YMC,EQTMAX
   41 FORMAT(//2X,'POISSONS RATIO FOR FILAMENT=',F10.5//2X,'POISSONS RAT
     1IO FOR MATRIX=',F10.5//2X,'YOUNGS MODULUS FOR FILAMENT=',E17.8//2X
     2,'YOUNGS MODULUS FOR MATRIX=',E17.8//2X,'YIELD LIMIT FOR FILAMENT=
     3',E17.8//2X,'YIELD LIMIT FOR MATRIX(TENSION)=',E17.8//2X,'YIELD LI
     4MIT FOR MATRIX(COMP.)=',E17.8//2X,'ULTIMATE STRESS FOR MATRIX=',E1
     57.8)
C
      READ(5,50) ZNF,BF,ZNMT,BMT,ZNMC,BMC,RYMAX,TEST
   50 FORMAT(8F10.5)
      WRITE(6,51)
   51 FORMAT(//2X,'COEFFICIENTS FOR POWER PLASTIC STRESS-STRAIN LAW:
     1E=(S/B)**N',/12X,'FILAMENT', 8X,'MATRIX(TENSION)', 8X,'MATRIX(COMP
     2.)')
      WRITE(6,52) ZNF,ZNMT,ZNMC,BF,BMT,BMC
   52 FORMAT(/3X,'N',3F18.5/3X,'B',3E18.4)
      WRITE(6,53) RYMAX,TEST
   53 FORMAT(//2X,'MAXIMUM STEP=',F10.5//2X,'TEST=',F10.5)
      READ(5,10) (NCP(I),I=1,NTCP)
      WRITE(6,54)
   54 FORMAT(//2X,'CORNER NODAL POINTS')
      WRITE(6,55) (I,NCP(I),I=1,NTCP)
   55 FORMAT(2I10)
C
              IF(NAL.EQ.0) GO TO 5
      READ (5,60) (NOLO(I),PV(I),I=1,NAL)
   60 FORMAT(I10,F10.1)
      WRITE(6,61)
   61 FORMAT(//2X,'APPLIED LOADS')
      WRITE (6,65) (I,NOLO(I),PV(I),I=1,NAL)
   65 FORMAT (5X,'APPLIED LOAD NO. ',I5,5X,'DEGREE OF FREEDOM ',I5,5X,'L
     1OAD =',F10.1)
C
    5 READ (5,10) (NBDY(I),I=1,NBC)
      WRITE(6,71)
   71 FORMAT(//2X,'ZEROED BOUNDARY DISPLACEMENTS')
      WRITE (6,75) (I, NBDY(I), I=1,NBC)
   75 FORMAT (5X,'BOUNDARY CONDITION NO.',I5,5X,'SUPPRESSED DOF',I5)
C
              IF(NUBC.EQ.0) GO TO 100
              READ(5,80) (NUDISP(I),UVC(I),I=1,NUBC)
   80         FORMAT(I10,F10.5)
      WRITE(6,81)
   81 FORMAT(//2X,'NON-ZERO BOUNDARY DISPLACEMENTS')
      WRITE(6,85) (I,NUDISP(I),UVC(I),I=1,NUBC)
   85 FORMAT (5X,'BOUNDARY CONDITION NO.',I5,5X,'SUPPRESSED DOF',I5,5X,
```

```
     1'DISPLACEMENT=',F10.5)                                            MA1450
C                                                                       MA1460
C                                                                       MA1470
C   *****************************************************************  MA1480
C    NOTATION AND DEFINITIONS                                           MA1490
C   *****************************************************************  MA1500
C                                                                       MA1510
C                                                                       MA1520
C    A( )=TRIANGULAR AREA                                               MA1530
C    AVESYZ=MACRCSCOPIC YZ SHEAR STRESS                                 MA1540
C    AVESXZ=MACRGSCOPIC XZ SHEAR STRESS                                 MA1550
C    BM=POWER LAW COFFICIENT(MATRIX)                                    MA1560
C    BMC=POWER LAW COFFICIENT IN COMPRESSION(MATRIX)                    MA1570
C    BMT=POWER LAW COFFICIENT IN TENSION(MATRIX)                        MA1580
C    CP(IX,2,2)=STRESS-STRAIN MATRIX FOR LONGITUDINAL SHEAR             MA1590
C    DELEQT=INCREMENTAL EQUIVALENT STRESS                               MA1600
C    CELEXZ=CHANGE IN XZ STRAIN                                         MA1610
C    CELEYZ=CHANGE IN YZ STRAIN                                         MA1620
C    DELTXZ=CHANGE IN XZ STRESS                                         MA1630
C    DELTYZ=CHANGE IN YZ STRESS                                         MA1640
C    DLPEXZ=CHANGE IN PLASTIC XZ STRAIN                                 MA1650
C    DLPEYZ=CHANGE IN PLASTIC YZ STRAIN                                 MA1660
C    EF=YOUNG'S MODULUS FOR THE FILAMENT                                MA1670
C    EM=YOUNG'S MODULUS FOR THE MATRIX                                  MA1680
C    E=ENERGY                                                           MA1690
C    EQS=EQUIVALENT STRAIN                                              MA1700
C    EQSTRN=CHANGE IN EQUIVALENT STRAIN                                 MA1710
C    EQT=EQUIVALENT STRESS                                              MA1720
C    EQTMAX=ULTIMATE TENSILE.STRESS                                     MA1730
C    EXZ=XZ STRAIN                                                      MA1740
C    EYZ=YZ STRAIN                                                      MA1750
C    GF=SHEAR MODULUS FOR THE FILAMENT                                  MA1760
C    GM=SHEAR MODULUS FOR THE MATRIX                                    MA1770
C    GRAND=SYSTEM STIFFNESS MATRIX                                      MA1780
C    GRANK=BANDED SYSTEM STIFFNESS MATRIX                               MA1790
C    HPR=SLOPE OF THE STRESS-STRAIN CURVE IN THE PLASTIC RANGE          MA1800
C    IMNNP=INITIAL MATRIX NODAL POINT                                   MA1810
C    NAL=NUMBER OF APPLIED LOADS                                        MA1820
C    NBAN=BANDWIDTH                                                     MA1830
C    NBC=NUMBER OF ZERP BOUNDARY COUNDITIONS                            MA1840
C    NBPX=NUMBER OF BOUNDARY POINTS ON X FACE                           MA1850
C    NBPY=NUMBER OF BOUNDARY POINTS ON Y FACE                           MA1860
C    NDFNP=NUMBER OF DEGREES OF FREEDOM AT EACH NCDAL POINT             MA1870
C    NELBX=NUMBER OF ELEMENTS WITH AN EDGE ON THE X FACE                MA1880
C    NELBY=NUMBER OF ELEMENTS WITH AN EDGE ON THE Y FACE                MA1890
C    NFLEL=NUMBER OF FILAMENT ELEMENTS                                  MA1900
C    NFNP=NUMBER OF FILAMENT NODAL POINTS                               MA1910
C    NINC=MAXIMUM NUMBER OF PLASTIC STEPS                               MA1920
```

```
C     NL=CYCLE COUNTER
C     NPEL( )=IDENTIFIES THE ITH PLASTIC ELEMENT
C     NMTEL=NUMBER OF MATRIX ELEMENTS
C     NMYEL=NUMBER OF YIELD ELEMENTS
C     NOLO( )=DEGREES OF FREEDOM
C     NTCP=NUMBER OF TRIANGLE CORNER POINTS
C     NUBC=NUMBER OF NON ZERO BOUNDARY CONDITIONS
C     NUDISP=SUPPRESSED DEGREES OF FREEDOM
C     NUMDF=NUMBER OF DOF
C     NUMEL=NUMBER OF ELEMENTS
C     NUMNP=NUMBER OF NODAL POINTS
C     P( )=APLIED LOAD
C     POISF=POISSON'S RATIO FOR THE FILAMENT
C     POISM=POISSON'S RATIO FOR THE MATRIX
C     REMIN=SCALE FACTOR TO BRING FIRST ELEMENT TO INITIAL YIELD
C     RYMAX=MAXIMUM STEP
C     RYMIN=SCALE FACTOR FOR INCREMENTAL STEP FOR A POST YIELD CYCLE
C     SII=DILATIONAL STRES (EQUALS ZERO FOR THE SHEAR PROBLEM)
C     SXZ=DEVIATORIC XZ SHEAR STRESS (= TAUXZ FOR THE SHEAR PROBLEM)
C     SYZ=DEVIATORIC YZ SHEAR STRESS (=TAUYZ FOR THE SHEAR PROBLEM)
C     T( )=ELEMENT THICKNESS
C     TAUXZ=XZ STRESS
C     TAUYZ=YZ STRESS
C     TEST=FACTOR OF CLOSENESS, SUCH THAT ELEMENT IS CONSIDERED AT YIELD
C     TOPLSN( )=TOTAL PLASTIC EQUIVALENT STRAIN
C     UVC=DISPLACEMENT
C     W( )=OUT OF PLANE DISPLACEMENT
C     X(1,J)=X-COORDINATE
C     X(2,J)=Y-COORDINATE
C     YF=YIELD FOR THE FILAMENT
C     YM=YIELD FOR THE MATRIX
C     YMC=YIELD FOR THE MATRIX IN COMPRESSION
C     YMT=YIELD FOR THE MATRIX IN TENSION
C     ZNM=POWER LAW EXPONENT (MATRIX)
C     ZNMC=POWER LAW EXPONENT IN COMPRESSION (MATRIX)
C     ZNMT=POWER LAW EXPONENT IN TENSION (MATRIX)
C
C
C    ********************************************************************
C
C
  100 CONTINUE
C
C    ********************************************************************
C FORMAT STATEMENTS
C    ********************************************************************
C
   90 FORMAT(5X,'SYSTEM NODAL POINT NO. =',I5,5X,'X(1) COORD. =',F10.5,
```

```
     15X,'X(2) COORD. =',F10.5)
  172 FORMAT(I10)
 1202 FORMAT(2I10,F10.5)
 1282 FORMAT(1H1/(I20,D20.8))
 2402 FORMAT(5X,2I10,3D17.8)
 2998 FORMAT(10X,'SXZ=',E17.8,10X,'SYZ=',E17.8,10X,'EQT=',E17.8)
 3003 FORMAT(/5X,2I10)
 3002 FORMAT(5X,3I10)
 3004 FORMAT(//2X,'STRAINS AND STRESSES AT INITIAL YIELD, TRIANGLE ',I5
     1/10X,'EXZ=',E17.8,10X,'EYZ=',E17.8,10X,'EQS=',E17.8)
 3102 FORMAT(1H1,5X,'CYCLE  ',I10)
 3222 FORMAT(5X,2I10,2E20.8)
 3252 FORMAT(//2X,'THE NEXT PLASTIC ELEMENT WILL BE',I5,5X,'RYMIM=',F10.
     15,5X,'RYMIN=',F10.5)
 3333 FORMAT(2X,6E17.8)
 3402 FORMAT(5X,3I10)
 3411 FORMAT(//2X,'STRAINS AND STRESSES FOR TRIANGLE',I5)
 3412 FORMAT(10X,'EXZ=',E17.8,10X,'EYZ=',E17.8,10X,'EQS=',E17.8)
 3413 FORMAT(10X,'SXZ=',E17.8,10X,'SYZ=',E17.8,10X,'EQT=',E17.8)
 3452 FORMAT(//2X,'MATRIX ELEMENT',I3,2X,'IS AT FAILURE')
 3502 FORMAT(1H05X,'THERE IS A NEGATIVE DELEP SINCE NP=',I5)
 4302 FCRMAT(5X,2I10,4E20.8)
 9999 FORMAT(18A4)
C
C
C   ***************************************************************
C
C
C     FOR THE ELASTIC ANALYSIS: KW=F (MA  2750-6860)
C
C
C   ***************************************************************
C
C
      NMTEL=NUMEL-NFLEL
C
C     INITIALIZE SOME COUNTERS   -
C
      NCR=0
      NL=1
      NBAND=0
      NEL=NFLEL
      NVAR=NUMEL
      IF(EM.GT.EF) NEL=1
      IF(EM.GT.EF) NVAR=NFLEL+1
      IF(EM.GT.EF) WRITE(6,95)
   95 FORMAT(//3X,'***COMPOSITE WITH SOFT FILAMENT AND STIFF MATRIX***')
C
```

```
C     ZERO OUT THE GRANK MATRIX
C
  160 DO 170 I=1,NUMDF
      DO 170 J=1,NBAN
      GRANK(I,J)=0.
      GRAND(I,J)=0.
C
  170 CONTINUE
C
C   OBTAIN COORDINATES OF TRIANGLE CORNER POINTS
C
      WRITE(6,89)
   89 FORMAT(//2X,'LOCATION OF NODAL POINTS')
      DO 180 I=1,NTCP
      J=NCP(I)
      X(1,J)=Y(1,I)
      X(2,J)=Y(2,I)
      WRITE(6,90) J,X(1,J),X(2,J)
C
  180 CONTINUE
C
      NBAND=0
C
C   *******************************************************************
C    TRIANGLE DO LOOP FOR ELASTIC ELEMENT STIFFNESS MATRICES
C   *******************************************************************
C
      ZETA(1,1)=2.
      ZETA(1,2)=1.
      ZETA(1,3)=1.
      ZETA(2,1)=1.
      ZETA(2,2)=2.
      ZETA(2,3)=1.
      ZETA(3,1)=1.
      ZETA(3,2)=1.
      ZETA(3,3)=2.
      GM=EM/(2.*(1.+POISM))
      GF=EF/(2.*(1.+POISF))
C
      DO 1000 IX=1,NUMEL
C     WRITE(6,172) IX
      G(IX)=GF
      IF(IX.GT.NFLEL) G(IX)=GM
      CP(IX,1,1)=G(IX)
      CP(IX,1,2)=0.
      CP(IX,2,1)=0.
      CP(IX,2,2)=G(IX)
      TOPLSN(IX)=0.
```

```
C
C    COMPUTE GEOMETRIC QUANTITIES
C
      JJ=ITEM(IX,1)
      KK=ITEM(IX,2)
      LL=ITEM(IX,3)
C
      A1=X(1,LL)-X(1,KK)
      A2=X(1,JJ)-X(1,LL)
      A3=X(1,KK)-X(1,JJ)
      B1=X(2,KK)-X(2,LL)
      B2=X(2,LL)-X(2,JJ)
      B3=X(2,JJ)-X(2,KK)
C
      A(IX)=(A3*B2-A2*B3)/2.
C
C    FORM THE 6X6   ELEMENT STIFFNESS MATRIX, SAY ZK
C
      DO 350  I=1,3
      DO 350  J=1,6
      ZX(IX,I,J)=0.
      ZY(IX,I,J)=0.
C
  350 CONTINUE
C
      ZX(IX,1,1)=3.*B1
      ZX(IX,1,2)=-B2
      ZX(IX,1,3)=-B3
      ZX(IX,1,4)=4.*B2
      ZX(IX,1,6)=4.*B3
      ZX(IX,2,1)=-B1
      ZX(IX,2,2)=3.*B2
      ZX(IX,2,3)=-B3
      ZX(IX,2,4)=4.*B1
      ZX(IX,2,5)=4.*B3
      ZX(IX,3,1)=-B1
      ZX(IX,3,2)=-B2
      ZX(IX,3,3)=3.*B3
      ZX(IX,3,5)=4.*B2
      ZX(IX,3,6)=4.*B1
      ZY(IX,1,1)=3.*A1
      ZY(IX,1,2)=-A2
      ZY(IX,1,3)=-A3
      ZY(IX,1,4)=4.*A2
      ZY(IX,1,6)=4.*A3
      ZY(IX,2,1)=-A1
      ZY(IX,2,2)=3.*A2
      ZY(IX,2,3)=-A3
```

```
      ZY(IX,2,4)=4.*A1
      ZY(IX,2,5)=4.*A3
      ZY(IX,3,1)=-A1
      ZY(IX,3,2)=-A2
      ZY(IX,3,3)=3.*A3
      ZY(IX,3,5)=4.*A2
      ZY(IX,3,6)=4.*A1
      DO 400  I=1,6
      DO 400  J=1,6
      ZK(I,J)=0.
      DO 390  K=1,3
      DC 390  L=1,3
      ZK(I,J)=ZK(I,J)+ZX(IX,K,I)*ZETA(K,L)*ZX(IX,L,J)
     1+ZY(IX,K,I)*ZETA(K,L)*ZY(IX,L,J)
C
  390 CONTINUE
C
      ZK(I,J)=G(IX)*T(IX)*ZK(I,J)/(96.*A(IX))
C
  400 CCNTINUE
C
C     WRITE(6,3333) ((ZK(I,J),J=1,6),I=1,6)
C
C   ******************************************************************
C    ASSEMBLY OF THE SYSTEM GRANK MATRIX
C   ******************************************************************
C
      DC 995  J=1,6
      II=ITEM(IX,J)
      DO 990  K=1,6
      III=ITEM(IX,K)
      IF(II.GT.III) GO TO 990
      NW=III-II+1
      IF(NW.GT.NBAND) NBAND=NW
      GRANK(II,NW)=GRANK(II,NW)+ZK(J,K)
      GRAND(II,NW)=GRANK(II,NW)
C
  990 CONTINUE
  995 CCNTINUE
 1000 CONTINUE
C
      IF(NL.EQ.1) WRITE(6,997) NBAND
  997 FORMAT(/5X,'NBAND=',I5)
      IF(NBAN.NE.NBAND) WRITE(6,998)
  998 FORMAT(5X,'ERROR IN INPUT OF NBAN SO PROGRAM IS TERMINATED')
      IF(NBAN.NE.NBAND) GO TO 9000
C
 1010 CONTINUE
```

```
C
C     WRITE(6,3333) ((GRANK(I,J),J=1,6),I=1,6)
C
C
C  ********************************************************************
C
C
C    1.) MODIFY GRANK TO ACCOMMODATE LOADS AND BOUNDARY CONDITIONS.
C    CALCULATE THE RESULTING DISPLACEMENTS, STRAINS AND STRESSES FOR THE
C    ELASTIC BEHAVIOR (SYSTEM STIFFNESS USES THE ELASTIC STRESS-STRAIN
C    RELATION HERE). (MA  4570-5930)
C
C
C    5.)  CHOOSE A TEST INCREMENT OF DISPLACEMENT, FOR THE START OF
C    THE PLASTIC CYCLE K(W) DW = DF .
C
C
C  ********************************************************************
C
C
C  ********************************************************************
C INPUT THE APPLIED LOAD VECTOR
C  ********************************************************************
C
 1050 DC 1100 I=1,NUMDF
 110C P(I)=0.
              IF(NAL.EQ.0) GO TO 1160
      DO 1150 I=1,NAL
      JJ=NCLO(I)
      P(JJ)=PV(I)
C
 1150 CONTINUE
C
C  ********************************************************************
C REVISE GRANK TO ACCOMODATE BOUNDARY CONDITIONS
C  ********************************************************************
C
 1160 DO 1190 I=1,NBC
      JJ=NBDY(I)
      DO 1180 J=2,NBAN
      KK=JJ-J+1
      GRANK(JJ,J)=0.
      IF (KK.LT.1) GO TO 1180
      GRANK(KK,J)=0.
C
 1180 CONTINUE
C
      GRANK(JJ,1)=1.
```

```
      P(JJ)=0.                                                          MA4810
C                                                                       MA4820
 1190 CONTINUE                                                          MA4830
C                                                                       MA4840
C   *****************************************************************  MA4850
C   REVISE GRANK TO ACCOMODATE NON ZERO DISPLACEMENT B.C.'S            MA4860
C   *****************************************************************  MA4870
C                                                                       MA4880
              IF(NUBC.EQ.0) GO TO 1280                                  MA4890
              DO 1260 I=1,NUBC                                          MA4900
      UBC(I)=UVC(I)                                                     MA4910
              JJ=NUDISP(I)                                              MA4920
C             WRITE(6,1202) I,JJ,UBC(I)                                 MA4930
              JJJ=JJ-NBAN+1                                             MA4940
              IF(JJJ.GT.1) GO TO 1200                                   MA4950
              JJJ=1                                                     MA4960
 1200         DO 1210 J=JJJ,JJ                                          MA4970
              K=JJ-J+1                                                  MA4980
              P(J)=P(J)-GRANK(J,K)*UBC(I)                               MA4990
 1210         CONTINUE                                                  MA5000
              KK=JJ+1                                                   MA5010
              KKK=JJ+NBAN-1                                             MA5020
              IF(KKK.LT.NUMDF) GO TO 1220                               MA5030
              KKK=NUMDF                                                 MA5040
      IF(KK.GT.KKK) GO TO 1235                                          MA5050
 1220         DO 1230 J=KK,KKK                                          MA5060
              JJJ=J-JJ+1                                                MA5070
              P(J)=P(J)-GRANK(JJ,JJJ)*UBC(I)                            MA5080
C                                                                       MA5090
 1230         CONTINUE                                                  MA5100
 1235 CONTINUE                                                          MA5110
C                                                                       MA5120
              DO 1240 M=1,NBAN                                          MA5130
              KK=JJ-M+1                                                 MA5140
              GRANK(JJ,M)=0.                                            MA5150
              IF(KK.LT.1) GO TO 1240                                    MA5160
              GRANK(KK,M)=0.                                            MA5170
C                                                                       MA5180
 1240         CONTINUE                                                  MA5190
C                                                                       MA5200
              GRANK(JJ,1)=1.                                            MA5210
              P(JJ)=UBC(I)                                              MA5220
C                                                                       MA5230
 1260         CONTINUE                                                  MA5240
C                                                                       MA5250
C             WRITE(6,1282) (I,P(I),I=1,NUMDF)                          MA5260
C                                                                       MA5270
 1280 CONTINUE                                                          MA5280
```

```
C                                                                       MA5290
C                                                                       MA5300
C   ******************************************************************  MA5310
C                                                                       MA5320
C                                                                       MA5330
C     6.)  SOLVE THE DISPLACEMENT-FORCE EQUATION FOR THE DISPLACEMENT   MA5340
C     AND THEN CALCULATE THE STRAIN INCREMENT AND THE STRESS INCREMENT  MA5350
C     (MA  5430-5800)                                                   MA5360
C                                                                       MA5370
C                                                                       MA5380
C   ******************************************************************  MA5390
C                                                                       MA5400
C                                                                       MA5410
C   ******************************************************************  MA5420
      CALL BANDEC(GRANK,P,NUMDF,NBAN,1,215,51 ,1)                       MA5430
C   ******************************************************************  MA5440
C                                                                       MA5450
C PRINT OUTPUT                                                          MA5460
C                                                                       MA5470
C     WRITE (6,1282) (I,P(I), I=1,NUMDF)                                MA5480
              DO 1290 I=1,NUMDF                                         MA5490
              UE(I)=P(I)                                                MA5500
C                                                                       MA5510
 1290         CONTINUE                                                  MA5520
C                                                                       MA5530
      DC 1295 I=1,NUMDF                                                 MA5540
      P(I)=0.                                                           MA5550
C                                                                       MA5560
 1295 CCNTINUE                                                          MA5570
C                                                                       MA5580
C   ******************************************************************  MA5590
      CALL MULI(NUMDF,NBAN,P,GRAND,UE)                                  MA5600
C   ******************************************************************  MA5610
C                                                                       MA5620
C     WRITE(6,1282) (I,P(I),I=1,NUMDF)                                  MA5630
C                                                                       MA5640
C    CALCULATE THE ENERGY E= F*U                                        MA5650
C                                                                       MA5660
      E=0.                                                              MA5670
      DO 1350 I=1,NUMDF                                                 MA5680
      E=E+P(I)*UE(I)                                                    MA5690
C                                                                       MA5700
 1350 CONTINUE                                                          MA5710
C                                                                       MA5720
      WRITE(6,1352) E                                                   MA5730
 1352 FORMAT(5X,'ENERGY=',E20.8)                                        MA5740
      IF(NL.NE.1) GO TO 3010                                            MA5750
C                                                                       MA5760
```

```
C     DETERMINE THE FIRST ELEMENT TO REACH YIELD.                       MA5770
C                                                                       MA5780
C   ******************************************************************  MA5790
      CALL CALTAU(TAUXZ,TAUYZ,SXZ,SYZ,EXZ,EYZ,NUMEL)                    MA5800
C   ******************************************************************  MA5810
C                                                                       MA5820
C     DETERMINE THE ELEMENT WITH THE MAXIMUM EQUIVALENT STRESS.         MA5830
C                                                                       MA5840
C     CALCULATE THE AVERAGE SXZ                                         MA5850
C                                                                       MA5860
C      WRITE(10) ((TAUXZ(I),TAUYZ(I),I=1,NUMEL),AVESXZ)                 MA5870
C                                                                       MA5880
C     DETERMINE THE YIELD CRITERION ACCORDING TO THE SIGN OF THE HYDRO- MA5890
C     STATIC PRESSURE FOR MATERIALS WITH DIFFERENT TENSION AND COM-     MA5900
C     PRESSION YIELD CURVES.                                            MA5910
C                                                                       MA5920
      DO 2500 IX=1,NUMEL                                                MA5930
      SII(IX)=0.                                                        MA5940
      YM=YMT                                                            MA5950
      IF(SII(IX).LT.0.) YM=YMC                                          MA5960
      EQT(IX)=SQRT(3.*((SXZ(IX)**2)+(SYZ(IX)**2)))                      MA5970
      IF(IX.LE.NFLEL) RE=YF/EQT(IX)                                     MA5980
      IF(IX.EQ.1.) REMIN=RE                                             MA5990
      IF(IX.GT.NFLEL) RE=YM/EQT(IX)                                     MA6000
      IF(RE.GT.REMIN) GO TO 2400                                        MA6010
      REMIN=RE                                                          MA6020
      IYEL=IX                                                           MA6030
C                                                                       MA6040
 2400 CONTINUE                                                          MA6050
C                                                                       MA6060
C     WRITE(6,2402) IX,IYEL,RE,REMIN,EQT(IX)                            MA6070
C                                                                       MA6080
 2500 CONTINUE                                                          MA6090
C                                                                       MA6100
C                                                                       MA6110
C   ******************************************************************  MA6120
C                                                                       MA6130
C                                                                       MA6140
C     2.)  SCALE THE ELASTIC VALUES SO THAT THE ELEMENT WITH THE MAXIMUM MA6150
C     EQUIVALENT STRESS IS AT INITIAL YIELD,I.E. ADJUSTMENT OF INPUT    MA6160
C     DISPLACEMENT SO THAT A SINGLE POINT IS AT YIELD.  (MA  6230-6860) MA6170
C                                                                       MA6180
C                                                                       MA6190
C   ******************************************************************  MA6200
C                                                                       MA6210
C                                                                       MA6220
      NMYEL=0                                                           MA6230
      TYF=TEST*YF                                                       MA6240
```

```
      TYM=TEST*YMT                                                      MA6250
      DC 3000 IX=1,NUMEL                                                MA6260
      TEQT=TYF                                                          MA6270
      IF(IX.GT.NFLEL.AND.SII(IX).GE.0.) TEQT=TEST*YMT                   MA6280
      IF(IX.GT.NFLEL.AND.SII(IX).LT.0.) TEQT=TEST*YMC                   MA6290
      EXZ(IX)=REMIN*EXZ(IX)                                             MA6300
      EYZ(IX)=REMIN*EYZ(IX)                                             MA6310
      TAUXZ(IX)=REMIN*TAUXZ(IX)                                         MA6320
      TAUYZ(IX)=REMIN*TAUYZ(IX)                                         MA6330
      DLPEXZ(IX)=    EXZ(IX)-TAUXZ(IX)/G(IX)                            MA6340
      DLPEYZ(IX)=    EYZ(IX)-TAUYZ(IX)/G(IX)                            MA6350
      IF(DLPEXZ(IX).EQ.0.AND.DLPEYZ(IX).EQ.0.) EQSTRN=0.                MA6360
      IF(DLPEXZ(IX).EQ.0.AND.DLPEYZ(IX).EQ.0.) GC TO 2510               MA6370
      EQSTRN=SQRT(((DLPEXZ(IX)**2)+(DLPEYZ(IX)**2))/3.)                 MA6380
 2510 TCPLSN(IX)=TOPLSN(IX)+EQSTRN                                      MA6390
      WRITE(6,3004) IX,EXZ(IX),EYZ(IX),TOPLSN(IX)                       MA6400
      SXZ(IX)=REMIN*SXZ(IX)                                             MA6410
      SYZ(IX)=REMIN*SYZ(IX)                                             MA6420
      EQT(IX)=REMIN*EQT(IX)                                             MA6430
      WRITE(6,2998) SXZ(IX),SYZ(IX),EQT(IX)                             MA6440
      SII(IX)=REMIN*SII(IX)                                             MA6450
      IF(EQT(IX).LE.TEQT) GO TO 3000                                    MA6460
      WRITE(6,3401)                                                     MA6470
      NMYEL=NMYEL+1                                                     MA6480
      NPEL(NMYEL)=IX                                                    MA6490
C     WRITE(6,3002) NMYEL,IX,NPEL(NMYEL)                                MA6500
C                                                                       MA6510
 3000 CONTINUE                                                          MA6520
C                                                                       MA6530
      WRITE(6,3008)                                                     MA6540
      WRITE(6,3003) (I,NPEL(I),I=1,NMYEL)                               MA6550
 3008 FORMAT(//5X,'PLASTIC ELEMENTS AT INITIAL YIELD'/)                 MA6560
C                                                                       MA6570
C                                                                       MA6580
C  ****************************************************************     MA6590
C                                                                       MA6600
C                                                                       MA6610
C    THIS TERMINATES THE ELASTIC ANALYSIS, ALTHOUGH SOME PCRTICNS OF THE MA6620
C    COMPUTER PROGRAM OVERLAP FOR THE PLASTIC AND ELASTIC ANALYSIS.     MA6630
C    FOR THE PLASTIC ANALYSIS THAT FOLLOWS WE PRCCEED IN CYCLES OR      MA6640
C    INCREMENTS.  THE PROCEDURE FOR THE PLASTIC ANALYSIS IS             MA6650
C    CYCLE AND CONTINUES, UNTIL ONE POINT (ELEMENT) OF THE              MA6660
C    COMPOSITE HAS FAILED.  THE LOADING FOR THE PLASTIC PROBLEM MAY     MA6670
C    MEAN EITHER DISPLACEMENT OR FORCE LOADING DEPENDING ON THE         MA6680
C    PARTICULAR PROBLEM.                                                MA6690
C                                                                       MA6700
C                                                                       MA6710
C    3.)  CALCULATE THE STIFFNESS MATRICES FOR THE POST YIELD ELEMENTS; MA6720
```

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C     USUALLY CNLY ONE ELEMENT WILL REACH YIELD IN THE FIRST CYCLE.       MA6730
C     HOWEVER SINCE ELEMENTS CLOSE TO YIELD WILL BE CARRIED INTO YIELD    MA6740
C     DURING THE NEXT INCREMENT OF CISPLACEMENT, ANY ELEMENT THAT IS      MA6750
C     WITHIN A "TEST" VALUE OF YIELD IS CONSIDEREC TO BE AT YIELD         MA6760
C     THE VALUE "TEST" IS COMPLETELY ARBITRARY; IT DEPENDS ON THE         MA6770
C     ACCURACY THAT THE USER WISHES TO CBTAIN. (MA 6840)                  MA6780
C                                                                         MA6790
C                                                                         MA6800
C   ******************************************************************   MA6810
C                                                                         MA6820
C                                                                         MA6830
      GO TO 3510                                                          MA6840
C                                                                         MA6850
 3010 CCNTINUE                                                            MA6860
C                                                                         MA6870
C   ******************************************************************   MA6880
      CALL CALTAU(DELTXZ,DELTYZ,DELSXZ,DELSYZ,DELEXZ,DELEYZ,NUMEL)        MA6890
C   ******************************************************************   MA6900
C                                                                         MA6910
      IYEL=0                                                              MA6920
      RYMIN=RYMAX                                                         MA6930
C                                                                         MA6940
C     THESE STEP CHANGES ARE ADJUSTED TO OBTAIN REASONABLE ACCURACY FOR  MA6950
C     THE PROBLEM UNDER CONSIDERATION                                     MA6960
C                                                                         MA6980
C                                                                         MA6990
C   ******************************************************************   MA7000
C                                                                         MA7010
C                                                                         MA7020
C     7.)  CALCULATE THE SCALE FACTOR FOR EACH REMAINING ELASTIC ELEMENT MA7030
C     TO REACH YIELD.  (MA  7100-7440)                                    MA7040
C                                                                         MA7050
C                                                                         MA7060
C   ******************************************************************   MA7070
C                                                                         MA7080
C                                                                         MA7090
      IF(NMYEL.EQ.NMTEL) NVAR=NFLEL                                       MA7100
      IF(NMYEL.GT.NMTEL) GO TO 3300                                       MA7110
 320C DC 3250 IX=NEL,NVAR                                                 MA7120
      IF(NMYEL.EQ.NMTEL) GO TO 3170                                       MA7130
      DC 3150 IXP=1,NMYEL                                                 MA7140
      JJ=NPEL(IXP)                                                        MA7150
      IF(JJ.EQ.IX) GO TO 3250                                             MA7160
C                                                                         MA7170
 3150 CONTINUE                                                            MA7180
C                                                                         MA7190
 3170 DELEQT(IX)=SQRT(3.*((DELTXZ(IX)**2)+(DELTYZ(IX)**2)))               MA7200
      GAMMA=-6*(DELTXZ(IX)*TAUXZ(IX)+DELTYZ(IX)*TAUYZ(IX))                MA7210
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      BEQT=DELEQT(IX)**2                                                MA7220
      GE= GAMMA**2+4.*BEQT*((YF**2)-(EQT(IX)**2))                       MA7230
      YM=YMT                                                            MA7240
      IF(SII(IX).LT.0.) YM=YMC                                          MA7250
      IF(IX.GT.NFLEL) GE=(GAMMA**2)+4.*BEQT*((YM**2)-(EQT(IX)**2))      MA7260
      RP=ABS((GAMMA+SQRT(GE))/(2.*BEQT))                                MA7270
      RM=ABS((GAMMA-SQRT(GE))/(2.*BEQT))                                MA7280
C                                                                       MA7290
C                                                                       MA7300
C   ******************************************************************* MA7310
C                                                                       MA7320
C                                                                       MA7330
C    8.)  DETERMINE THE MINIMUM SCALE FACTOR NECESSARY TO CAUSE THE     MA7340
C    NEXT ELEMENT TO REACH INITIAL YIELD. (MA  7530)                    MA7350
C                                                                       MA7360
C                                                                       MA7370
C   ******************************************************************* MA7380
C                                                                       MA7390
C                                                                       MA7400
      RY=AMIN1(RP,RM)                                                   MA7410
      IF(IX.EQ.NFLEL) RYMIM=RY                                          MA7420
      IF(RY.GT.RYMIM) GO TO 3220                                        MA7430
      RYMIM=RY                                                          MA7440
      IYEL=IX                                                           MA7450
C                                                                       MA7460
 3220 CONTINUE                                                          MA7470
C                                                                       MA7480
C     WRITE(6,3222) IX,IYEL,RY,RYMIM                                    MA7490
C                                                                       MA7500
 3250 CONTINUE                                                          MA7510
C                                                                       MA7520
      RYMIN=AMIN1(RYMIN,RYMIM)                                          MA7530
C                                                                       MA7540
 3300 CONTINUE                                                          MA7550
C                                                                       MA7560
      WRITE(6,3252) IYEL,RYMIM,RYMIN                                    MA7570
C                                                                       MA7580
C                                                                       MA7590
C   ******************************************************************* MA7600
C                                                                       MA7610
C                                                                       MA7620
C    9.)  MULTIPLY THE STRAIN INCREMENT AND STRESS INCREMENT CALCULATED MA7630
C    IN STEP SIX BY THE MINIMUM SCALE FACTOR AND ADD THE RESULTING DEL  MA7640
C    QUANTITIES AT THE NODES TO THE PREVIOUS NODAL VALUES.              MA7650
C    (MA  7720-7840)                                                    MA7660
C                                                                       MA7670
C                                                                       MA7680
C   ******************************************************************* MA7690
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C                                                                       MA7700
      NMYEL=0                                                           MA7710
      DO 3450 IX=1,NUMEL                                                MA7720
      WRITE(6,3411) IX                                                  MA7730
      DELEXZ(IX)=RYMIN*DELEXZ(IX)                                       MA7740
      DELEYZ(IX)=RYMIN*DELEYZ(IX)                                       MA7750
      EXZ(IX)=EXZ(IX)+DELEXZ(IX)                                        MA7760
      EYZ(IX)=EYZ(IX)+DELEYZ(IX)                                        MA7770
      DELTXZ(IX)=RYMIN*DELTXZ(IX)                                       MA7780
      DELTYZ(IX)=RYMIN*DELTYZ(IX)                                       MA7790
      TAUXZ(IX)=TAUXZ(IX)+DELTXZ(IX)                                    MA7800
      TAUYZ(IX)=TAUYZ(IX)+DELTYZ(IX)                                    MA7810
      DLPEXZ(IX)=DELEXZ(IX)-DELTXZ(IX)/G(IX)                            MA7820
      DLPEYZ(IX)=DELEYZ(IX)-DELTYZ(IX)/G(IX)                            MA7830
      IF(DLPEXZ(IX).EQ.0.AND.DLPEYZ(IX).EQ.0.) EQSTRN=0.                MA7840
      IF(DLPEXZ(IX).EQ.0.AND.DLPEYZ(IX).EQ.0.) GO TO 3380               MA7850
      EQSTRN=SQRT(((DLPEXZ(IX)**2)+(DLPEYZ(IX)**2))/3.)                 MA7860
 3380 TOPLSN(IX)=TOPLSN(IX)+EQSTRN                                      MA7870
      WRITE(6,3412) EXZ(IX),EYZ(IX),TOPLSN(IX)                          MA7880
      SXZ(IX)=SXZ(IX)+RYMIN*DELSXZ(IX)                                  MA7890
      SYZ(IX)=SYZ(IX)+RYMIN*DELSYZ(IX)                                  MA7900
C                                                                       MA7910
C                                                                       MA7920
C   ****************************************************************   MA7930
C                                                                       MA7940
C                                                                       MA7950
C    10.)  CALCULATE THE NEW EQUIVALENT STRESS FOR EACH ELEMENT.        MA7960
C    (MA 8040)                                                          MA7970
C                                                                       MA7980
C                                                                       MA7990
C   ****************************************************************   MA8000
C                                                                       MA8010
C                                                                       MA8020
      EQT(IX)=SQRT(3.*((SXZ(IX)**2)+(SYZ(IX)**2)))                      MA8030
      WRITE(6,3413) TAUXZ(IX),TAUYZ(IX),EQT(IX)                         MA8040
      TEQT=TYF                                                          MA8050
      IF(IX.GT.NFLEL) TEQT=TYM                                          MA8060
      IF(IX.GT.NFLEL.AND.SII(IX).LT.0.) TEQT=TEST*YMC                   MA8070
      IF(EQT(IX).LE.TEQT) GO TO 3440                                    MA8080
      NMYEL=NMYEL+1                                                     MA8090
      NPEL(NMYEL)=IX                                                    MA8100
      WRITE(6,3401)                                                     MA8110
 3401 FORMAT(5X,'***PLASTIC ELEMENT***')                                MA8120
C     WRITE(6,3402) NMYEL,IX,NPEL(NMYEL)                                MA8130
      IF(NFLEL.EQ.1) GO TO 3410                                         MA8140
      IF(EM.GT.EF.AND.IX.LE.NFLEL.AND.EQT(IX).GE.EQTMAX) GO TO 9000     MA8150
      IF(EM.GT.EF) GO TO 3410                                           MA8160
                                                                        MA8170
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      IF(IX.LE.NFLEL.AND.EQT(IX).GT.YF) GO TO 9000                        MA8180
 3410 IF(IX.GT.NFLEL.AND.EQT(IX).GE.EQTMAX) NCR=I                         MA8190
      IF(IX.GT.NFLEL.AND.EQT(IX).GE.EQTMAX) IFAIL=IX                      MA8200
C                                                                         MA8210
 3440 CONTINUE                                                            MA8220
 3450 CONTINUE                                                            MA8230
C                                                                         MA8240
      WRITE(6,3451) NMYEL                                                 MA8250
 3451 FORMAT(//2X,'THE NUMBER OF PLASTIC ELEMENTS IS',I5)                 MA8260
C                                                                         MA8270
C     TERMINATE THE ANALYSIS IF THE CYCLES ARE COMPLETED.                 MA8280
C                                                                         MA8290
C                                                                         MA8300
C   ******************************************************************   MA8310
C                                                                         MA8320
C                                                                         MA8330
C     11.)  CALCULATE THE INCREMENT OF PLASTIC STRAIN FOR EACH POST YIELD MA8340
C     ELEMENT.  THE STIFFNESSES OF EACH  PLASTIC ELEMENT CONSTANTLY       MA8350
C     CHANGES AS THE ELEMENT PROGRESSES INTO THE PLASTIC RANGE.           MA8360
C     (MA  7830-7840)                                                     MA8370
C                                                                         MA8380
C                                                                         MA8390
C   ******************************************************************   MA8400
C                                                                         MA8410
C                                                                         MA8420
      IF(NL.EQ.NINC) GO TO 9000                                           MA8430
      NP=0                                                                MA8440
      WRITE(6,4301)                                                       MA8450
 4301 FORMAT(//2X,'EQUIVALENT STRESS,EQUIVALENT STRAIN, DELEP AND HPRIME  MA8460
     1 FOR THE PLASTIC ELEMENTS')                                         MA8470
      DO 3500 IXP=1,NMYEL                                                 MA8480
      NIX=NPEL(IXP)                                                       MA8490
      E=EF                                                                MA8500
      BM=BMT                                                              MA8510
      ZNM=ZNMT                                                            MA8520
      IF(SII(IXP).LT.0.) BM=BMC                                           MA8530
      IF(SII(IXP).LT.0.) ZNM=ZNMC                                         MA8540
      IF(NIX.GT.NFLEL) E=EM                                               MA8550
      HPR(NIX)=(EQT(NIX)/ZNF)*((BF/EQT(NIX))**ZNF)                        MA8560
      IF(NIX.GT.NFLEL) HPR(NIX)=(EQT(NIX)/ZNM)*((BM/EQT(NIX))**ZNM)       MA8570
C     IF(NIX.GT.NFLEL.AND.EQT(NIX).GE.40.0E+03) HPR(NIX)=3.33E+05         MA8580
      GP=EF/(2*(1.+POISF))                                                MA8590
      IF(NIX.GT.NFLEL) GP=EM/(2*(1.+POISM))                               MA8600
      DENOM=EQT(NIX)*(1.0+HPR(NIX)/(3.*GP))                               MA8610
C                                                                         MA8620
C                                                                         MA8630
C   ******************************************************************   MA8640
C                                                                         MA8650
```

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C
C     12.)  CHECK THAT THE INCREMENTAL PLASTIC STRAIN IS POSITIVE.
C     IF IT IS POSITIVE RETURN TO STEP THREE.  IF IT IS NEGATIVE STOP
C     COMPUTATION.  (MA  8610-8760)
C
C
C   ******************************************************************
C
C
      DELEP(NIX)=((SXZ(NIX)*DELEXZ(NIX))+(SYZ(NIX)*DELEYZ(NIX)))/DENOM
      WRITE(6,4302) IXP,NIX,EQT(NIX),TOPLSN(NIX),DELEP(NIX),HPR(NIX)
      IF(DELEP(NIX).LT.1.E-20) NP=1
C
 3500 CONTINUE
C
      IF(NP.NE.1) GO TO 3510
      WRITE(6,3502) NP
      GC TO 9000
C
 3510 CONTINUE
C
      SXZ1=0.
      DO 2000 I=1,NELBX
      II=IELBX(I)
      SXZ1=SXZ1+TAUXZ(II)
C
 200C CONTINUE
C
      AVESXZ=SXZ1/NELBX
C
      SYZ1=0.
      DO 2010 I=1,NELBY
      II=IELBY(I)
      SYZ1=SYZ1+TAUYZ(II)
C
 2010 CONTINUE
C
      AVESYZ=SYZ1/NELBY
      WRITE(6,2002) AVESXZ,AVESYZ
 2002 FORMAT(//5X,'AVESXZ=',E17.8,10X,'AVESYZ=',E17.8)
C
      IF(NCR.EQ.1) WRITE(6,3452) IFAIL
      IF(NCR.EQ.1) GO TO 9000
C
C     CALCULATE POST YIELD ELEMENTS.
C
C   ******************************************************************
      CALL POST(POISF,POISM,EF,EM,ZNF,ZNMT,ZNMC,BF,BMT,BMC,
```

```
     1NMYEL,NFLEL,NUMDF,NBAN,NUMEL)
C   *****************************************************************
C
      NL=NL+1
      WRITE(6,3102) NL
      GO TO 1010
C
 900C CONTINUE
C
      STCP
      END
C     ENDFILE 10
C     END
```

```
C   *****************************************************************
C
      SUBROUTINE POST(POISF,POISM,EF,EM,ZNF,ZNMT,ZNMC,BF,BMT,BMC,
     1NMYEL,NFLEL,NUMDF,NBAN,NUMEL)
C
C   *****************************************************************
C
C    THIS SUBROUTINE MODIFIES THE STIFFNESS MATRIX
C
      COMMON/XYZ/A(92),SXZ(92),SYZ(92),ZX(92,3,6),ZY(92,3,6),CP(92,2,2),
     1ITEM(92,6)
      COMMON/XY/EQT(92),GRANK(215,51),GRAND(215,51),HPR(92),T(92),SII(92
     1),ZETA(3,3),NPEL(92)
C
      DIMENSION ZXX(6,6),ZYY(6,6),ZXY(6,6),ZZK(6,6),HPRIME(92)
C
      DO 6000 IXP=1,NMYEL
      NIX=NPEL(IXP)
      EV=EF/(1.+POISF)
      IF(NIX.GT.NFLEL) EV=EM/(1.+POISM)
      BM=BMT
      ZNM=ZNMT
      IF(SII(IXP).LT.0.) BM=BMC
      IF(SII(IXP).LT.0.) ZNM=ZNMC
      GG=0.5*EV
      HPR(NIX)=(EQT(NIX)/ZNF)*((BF/EQT(NIX))**ZNF)
      IF(NIX.GT.NFLEL) HPR(NIX)=(EQT(NIX)/ZNM)*((BM/EQT(NIX))**ZNM)
      IF(NIX.GT.NFLEL.AND.EQT(NIX).GE.40.0E+03) HPR(NIX)=3.33E+05
      S=(EQT(NIX)**2)*(1.+HPR(NIX)/(3.*GG))*(.6667)
      CP(NIX,1,1)=EV*(0.5-(SXZ(NIX)**2)/S)
      CP(NIX,1,2)=EV*(-SXZ(NIX)*SYZ(NIX)/S)
      CP(NIX,2,1)=CP(NIX,1,2)
      CP(NIX,2,2)=EV*(0.5-(SYZ(NIX)**2)/S)
      HPRIME(IXP)=HPR(NIX)
C
 6000 CONTINUE
C
      WRITE(6,3512) NMYEL
      WRITE(6,3513) (I,NPEL(I),HPRIME(I),I=1,NMYEL)
 3512 FORMAT(//2X,'THE NUMBER OF PLASTIC ELEMENTS=',I5)
 3513 FORMAT(2X,I5,5X,'ELEMENT',I5,10X,'HPR=',E17.8)
C
C     ZERO OUT THE GRANK MATRIX
C
  160 DO 170 I=1,NUMDF
      DO 170 J=1,NBAN
      GRANK(I,J)=0.
```

```
  170 CONTINUE
C
      DC 6050 NIX=1,NUMEL
C
C    ******************************************************************
C
C
C     4.)  MODIFY THE SYSTEM STIFFNESS MATRIX TO REFLECT THE STIFFNESS
C     OF THE ELEMENTS THAT HAVE YIELDED. (PO  0500-0990)
C
C
C    ******************************************************************
C
      DC 4500 I=1,6
      DO 4500 J=1,6
      ZZK(I,J)=0.
      ZXX(I,J)=0.
      ZXY(I,J)=0.
      ZYY(I,J)=0.
      DC 4000 K=1,3
      DO 4000 L=1,3
      ZXX(I,J)=ZXX(I,J)+ZX(NIX,K,I)*ZETA(K,L)*ZX(NIX,L,J)
      ZXY(I,J)=ZXY(I,J)+ZX(NIX,K,I)*ZETA(K,L)*ZY(NIX,L,J)
      ZYY(I,J)=ZYY(I,J)+ZY(NIX,K,I)*ZETA(K,L)*ZY(NIX,L,J)
C
 4000 CCNTINUE
C
      ZZK(I,J)=(CP(NIX,1,1)*ZXX(I,J)+2.*CP(NIX,1,2)*ZXY(I,J)+CP(NIX,2,2)
     1*ZYY(I,J))*T(NIX)/(96.*A(NIX))
C
 45CC CCNTINUE
C
      DO 4520 J=1,6
      II=ITEM(NIX,J)
      DO 4510 K=1,6
      III=ITEM(NIX,K)
      IF(II.GT.III) GO TO 4510
      NW=III-II+1
      IF(NW.GT.NBAN) NBAN=NW
      GRANK(II,NW)=GRANK(II,NW)+ZZK(J,K)
      GRAND(II,NW)=GRANK(II,NW)
C
 4510 CONTINUE
 4520 CCNTINUE
 6050 CCNTINUE
C     WRITE(6,3333) ((GRANK(I,J),J=1,6),I=1,6)
 3333 FORMAT(2X,6E17.8)
      RETURN
```

```
C
C
C   ******************************************************************
C
      SUBROUTINE CALTAU(TAUZX,TAUZY,SZX,SZY,EZX,EZY,NUMEL)
C
C   ******************************************************************
C
C    THIS SUBROUTINE CALCULATES THE STRESSES
C
C
      COMMON/XYZ/A(92),SXZ(92),SYZ(92),ZX(92,3,6),ZY(92,3,6),CP(92,2,2),
     1ITEM(92,6)
      COMMON/XZ/UE(215)
C
      DIMENSION W(6),EXZC(3),EYZC(3)
      DIMENSION TAUZX(92),TAUZY(92),SZX(92),SZY(92),EZX(92),EZY(92)
C
C    CALCULATION OF CENTROIDAL SHEAR STRAINS AND STRESSES
C
      DO 1600 IX=1,NUMEL
      II=ITEM(IX,1)
      JJ=ITEM(IX,2)
      KK=ITEM(IX,3)
      LL=ITEM(IX,4)
      MM=ITEM(IX,5)
      NN=ITEM(IX,6)
C
      W(1)=UE(II)
      W(2)=UE(JJ)
      W(3)=UE(KK)
      W(4)=UE(LL)
      W(5)=UE(MM)
      W(6)=UE(NN)
C
      DO 1450 I=1,3
      EXZC(I)=0.
      EYZC(I)=0.
      DO 1400 J=1,6
      EXZC(I)=EXZC(I)+ZX(IX,I,J)*W(J)
      EYZC(I)=EYZC(I)+ZY(IX,I,J)*W(J)
C
 1400 CONTINUE
C
      EXZC(I)=EXZC(I)/(4.*A(IX))
      EYZC(I)=EYZC(I)/(4.*A(IX))
C
```

```
 1450 CCNTINUE
C
      EZX(IX)=(EXZC(1)+EXZC(2)+EXZC(3))/3.
      EZY(IX)=(EYZC(1)+EYZC(2)+EYZC(3))/3.
      TAUZX(IX)=CP(IX,1,1)*EZX(IX)+CP(IX,1,2)*EZY(IX)
      TAUZY(IX)=CP(IX,1,2)*EZX(IX)+CP(IX,2,2)*EZY(IX)
      SZX(IX)=TAUZX(IX)
      SZY(IX)=TAUZY(IX)
C
C     WRITE(6,1602) IX,EXZ(IX),EYZ(IX),TAUXZ(IX),TAUYZ(IX)
 1602 FCRMAT(I10,4E20.8)
C
 1600 CONTINUE
C
      RETURN
      END
```

```
C                                                                         BA0020
C   ************************************************************          BA0030

      SUBROUTINE BANDEC(A,F,NEQ,MAXB,NVEC,NROW,NBAN,NBEC)                 BA0040
C                                                                         BA0045
C   ************************************************************          BA0050
C                                                                         BA0060
C    THIS SUBROUTINE SOLVES A SYSTEM OF LINEAR ALGEBRAIC EQUATIONS        BA0070
C    WITH THE SYMMETRIC MATRIX IN BANDED FORM.                            BA0080
C                                                                         BA0090
      DIMENSION A(NROW,NBAN),F(NROW,NBEC)                                 BA0130
C                                                                         BA0140
C     DECOMPOSE-NO PIVOTING.                                              BA0150
C                                                                         BA0160
      LOOP=NEQ-1                                                          BA0170
      DO 100 I=1,LOOP                                                     BA0180
      IF ( ABS(A(I,1))-1.E-30 ) 1000,1000,10                              BA0190
   10 M=I+1                                                               BA0200
      N= MINO(I+MAXB-1,NEQ)                                               BA0210
      DO 100 J=M,N                                                        BA0220
      L=J+2-M                                                             BA0230
      D=A(I,L)/A(I,1)                                                     BA0240
      DO 40 MM=1,NVEC                                                     BA0250
   40 F(J,MM)=F(J,MM)-D*F(I,MM)                                           BA0260
      MM= MINO(MAXB-L+1,NEQ-J+1)                                          BA0270
      DO 100 K=1,MM                                                       BA0280
      NN=L+K-1                                                            BA0290
  100 A(J,K)=A(J,K)-D*A(I,NN)                                             BA0300
C                                                                         BA0310
C     BACK SUBSTITUTE.                                                    BA0320
C                                                                         BA0330
      DO 199 I=1,NVEC                                                     BA0340
  199 F(NEQ,I)=F(NEQ,I)/A(NEQ,1)                                          BA0350
      DO 200 I=2,NEQ                                                      BA0360
      J=NEQ-I+1                                                           BA0370
      K= MINO(NEQ-J+1,MAXB)                                               BA0380
      DO 200 MM=1,NVEC                                                    BA0390
      DO 190 L=2,K                                                        BA0400
      M=J+L-1                                                             BA0410
  190 F(J,MM)=F(J,MM)-A(J,L)*F(M,MM)                                      BA0420
  200 F(J,MM)=F(J,MM)/A(J,1)                                              BA0430
C                                                                         BA0440
      RETURN                                                              BA0450
 1000 WRITE (6,1010) I                                                    BA0460
      STOP                                                                BA0470
 1010 FORMAT (29H1 ZERO DIAGONAL ELEMENT, ROW I4)                         BA0480
C                                                                         BA0490
```

```
C                                                                            MU0010
C                                                                            MU0020
C     ******************************************************************     MU0030

      SUBROUTINE MULI(NUMDF,NBAN,F,S,U)                                      MU0040
C                                                                            MU0045
C     ******************************************************************     MU0050
C                                                                            MU0060
C                                                                            MU0070
C     SUBROUTINE MULI MULTIPLIES A SYMMETRIC BANDED MATRIX WITH A            MU0080
C     VECTOR                                                                 MU0090
C                                                                            MU0100
C     IMPLICIT REAL*8(A-H,O-Z)                                               MU0110
C                                                                            MU0120
      DIMENSION F(215),S(215,51),U(215)                                      MU0130
C                                                                            MU0140
      DO 120 J=1,NUMDF                                                       MU0150
      M=MIN0(NBAN,NUMDF+1-J)                                                 MU0160
      DO 105 K=1,M                                                           MU0170
      L=J+K-1                                                                MU0180
      F(J)=F(J)+S(J,K)*U(L)                                                  MU0190
C                                                                            MU0200
  105 CONTINUE                                                               MU0210
C                                                                            MU0220
      IF(J.EQ.1) GO TO 120                                                   MU0230
      L=MIN0(J-1,NBAN-1)+1                                                   MU0240
      DO 110 K=2,L                                                           MU0250
      M=J-K+1                                                                MU0260
      F(J)=F(J)+S(M,K)*U(M)                                                  MU0270
C                                                                            MU0280
  110 CONTINUE                                                               MU0290
  120 CONTINUE                                                               MU0300
C                                                                            MU0310
      RETURN                                                                 MU0320
C                                                                            MU0330
```

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13. ABSTRACT

The failure point of a unidirectional composite subjected to longitudinal shear loading is calculated. The method of analysis is based on the finite element technique, the incremental plasticity relations of Prandtl-Reuss, and the von Mises yield criterion. The failure point for a boron-aluminum composite and a boron-epoxy composite were computed to determine the effect of the matrix material on the composite.

14. KEY WORDS	LINK A		LINK B		LINK C	
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